Remedial Lesson 1: Review of Calculus Techniques

Prepared by Adrian Sai-wah TAM (swtam3@ie.cuhk.edu.hk)

September 15, 2005

1 Definition of Differentiation

- Differentiation as a limit:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \qquad =$$

  and ________ is called the derivative of $f(x)$.

- If we think $f(x)$ is a ...
  1. ...is a curve, then $f'(x)$ is the ________ of the curve at the coordinate ___________
  2. ...is the displacement of a motion, and$x$ is time, then $f'(x)$ is the ___________
  3. ...is the velocity of a motion, and$x$ is time, then $f'(x)$ is the ______________
  4. ...is the quantity of something (e.g. $$ in bank), and$x$ is time, then $f'(x)$ is the ________________ of the quantity

- Sometimes, we may ______________ the derivative by

$$f'(x) = \frac{\Delta f}{\Delta x}$$

- Higher differenatials: If $y = f(x)$,

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} f(x) = f'(x) \\ \frac{d^2y}{dx^2} &= \frac{d^2}{dx^2} f(x) = \frac{d}{dx} f'(x) \\ f^{(n)}(x) &= \underbrace{\frac{d}{dx}\frac{d}{dx}\cdots\frac{d}{dx}}_{n} f(x) \\ &= \frac{d^n}{dx^n} f(x) \\ &= \frac{d}{dx} f^{(n-1)}(x) \end{aligned}$$

2 Techniques of Differentiation

- Formulae to remember:

$$\begin{aligned} \frac{d}{dx} x^n &= \\ \frac{d}{dx} \frac{1}{x^n} &= \qquad = \\ \frac{d}{dx} \sin x &= \end{aligned}$$

$$\frac{d}{dx}\cos x =$$
$$\frac{d}{dx}\tan x =$$
$$\frac{d}{dx}\cot x =$$
$$\frac{d}{dx}\sec x =$$
$$\frac{d}{dx}\csc x =$$

- Differentiation rules:

$$\frac{d}{dx}k = \qquad (k \text{ is constant})$$
$$\frac{d}{dx}kf(x) = \qquad\qquad k \text{ is constant})$$
$$\frac{d}{dx}[f(x) \pm g(x)] =$$
$$\frac{d}{dx}f(x)g(x) =$$
$$\frac{d}{dx}\frac{f(x)}{g(x)} = \underline{\hspace{4cm}}$$
$$\frac{d}{dx}f(g(x)) =$$

Exercises

1. Evaluate $\frac{d}{dx}x^3 = \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h}$

$$\begin{aligned}\frac{d}{dx}x^3 &= \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h}\\ &= \lim_{h\to 0}\frac{x^3 + \qquad\qquad\qquad - x^3}{h}\\ &= \lim_{h\to 0}(\qquad\qquad\qquad)\\ &= \end{aligned}$$

2. Evaluate the derivative for $2x^2 + 13x + 15$

3. Using the product rule and chain rule to find the derivative of $2x^2+13x+15=(2x+3)(x+5)$

4. Evaluate $\frac{d}{dx}(2x+3)^{99}$

$$\frac{d}{dx}(2x+3)^{99} = (\qquad)\left(\frac{d}{dx}\qquad\qquad\right)$$
$$=$$
$$=$$

5. Evaluate the following:

$$\frac{d}{dx}\frac{x^2+2x+1}{x^2-2x+1} = \frac{(x^2-2x+1)\frac{d}{dx}(x^2+2x+1)-(x^2+2x+1)\frac{d}{dx}(x^2-2x+1)}{(x^2-2x+1)^2}$$

= ________________________________

= __________________________

= ________________________

= ____________

=

6. Evaluate the following:

$$\frac{d}{dx}\frac{x^2+2x+1}{x^2-2x+1} = \frac{d}{dx}\frac{(x+1)^2}{(x-1)^2}$$

= ____________________________

= ______________________________

=

=

=

7. Evaluate the following:

$$\frac{d}{dx}\frac{x^2+2x+1}{x^2-2x+1} = \frac{d}{dx}\frac{(x+1)^2}{(x-1)^2} = \frac{d}{dx}\left[(x+1)^2(x-1)^{-2}\right]$$

$$=$$

$$=$$

$$=$$

$$=$$

$$=$$

8. Evaluate $\frac{d^2}{dx^2}\sqrt{x} = \frac{d}{dx}\left(\frac{d}{dx}\sqrt{x}\right)$

9. Evaluate $\frac{d}{dx}\left(3\sin^2(x^2)\right)$

3 Indefinite Integral

- Integration as the __________ function of differentiation
- Examples:

$$\int \qquad dx = x^n$$

$$\int \qquad dx = \frac{1}{x^n}$$

$$\int \qquad dx = \sin x$$

$$\int \qquad dx = \cos x$$

$$\int \qquad dx = \tan x$$

$$\int \qquad dx = \cot x$$

$$\int \qquad dx = \sec x$$

$$\int \qquad dx = \csc x$$

- There is a *constant of integration* in the result of indefinite integral

4 Definite Integral

- Definition: Riemann Sum
  - Adding many many very very small quantities together

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{x_1=a}^{x_n=b} f(x_k)(x_{k+1} - x_k)$$

  - Example: Finding area under the curve $y = f(x)$, for $a \leq x \leq b$
- Fundamental Theorem of Calculus

$$\int_a^b f(x)dx = \int f(x)dx\Big|_a^b$$

$$\int_a^b F'(x)dx = F(b) - F(a)$$

  - Example:

$$\begin{aligned}\int_0^1 x^2 dx &= \int x^2 dx\Big|_0^1 \\ &= \frac{1}{3}x^3\Big|_0^1 \\ &= \frac{1}{3}(1)^3 - \frac{1}{3}(0)^3 \\ &= \frac{1}{3}\end{aligned}$$

- Additional theorems to remember:

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^a f(x)dx = 0$$

Exercises:

1. $\int_4^{121} \sqrt{x}dx$

2. $\int_0^{4\pi} \cos x dx$

3. $\int_5^2 x^2 dx$

4. $\int_0^1 mx^n dx$

5. (Caution: Indefinite integral) $\int \sum_{k=1}^{n} B_k (kx+1)^k dx$

5 Bring-home Practices

5.1 Differentiation

1. Find $\frac{dy}{dx}$ for $y = \frac{\sin x}{x} + \frac{x}{\sin x}$
2. Find $\frac{dy}{dx}$ for $y = \frac{3x^5}{\sqrt[5]{x^2}} - \frac{7}{\sqrt[3]{x}} + 3\sqrt[7]{x^3}$
3. Find $\frac{dy}{dx}$ for $y = (x-1)(2x+1)(3-2x)$
4. Find $\frac{dy}{dx}$ for $y = \sqrt{\frac{1-x}{1+x}}$
5. Find $\frac{dy}{dx}$ for $y = \sin\left(\cos^2(x^3+x)\right)$
6. Find $\frac{dy}{dx}$ for $y = \frac{1}{x}\cos\frac{1}{x}$
7. Find $\frac{dy}{dx}$ for $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$
8. The slope of the tangent to the curve $y = ax^3 + bx$ at the point $(1,1)$ is -5. Calculate the values of a and b. With these values, find the coordinates of the points on the curve where the tangent is parallel to the x-axis.
9. (Caution: Implicit differentiation) Find the slopes at the points where $x = 2$ on the curve

$$17x^2 - 12xy + 8y^2 = 100$$

10. (Caution: Implicit differentiation) Find the slope of the tangent at the point $(\frac{7}{4}, 0)$ to the curve

$$2x^2y - 3y^2x - 4x + 5y + 7 = 0$$

11. Given $x^3 - 3axy + y^3 = b^3$, find $\frac{d^2y}{dx^2}$
12. Given $y = \frac{x^4}{144} + x^3 + 54x^2 + ax + b$ where a and b are constants. Find a if $\left(\frac{d^2y}{dx^2}\right)^2 = y$ for all values of x.
13. If $v = \frac{1}{r} + c$, show that $\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} = 0$

5.2 Integration

1. Evaluate $\int \frac{1}{\cos^2 x} dx$
2. Evaluate $\int \frac{ax}{b} dx$
3. Evaluate $\int \frac{1}{\sqrt{2gh}} dh$
4. Evaluate $\int (x + 3\cos x) dx$
5. Evaluate $\int (3x-2)(4x+3) dx$
6. Evaluate $\int x^2(5-x)^4 dx$

7. Evaluate $\int (2x+3)^3 dx + \int (2x-3)^3 dx$

8. Evaluate $\int (\sqrt{x}+1)(x-\sqrt{x}+1)dx$

9. Evaluate $\int \frac{\sqrt[3]{x^2}-\sqrt[4]{x}}{\sqrt{x}} dx$

10. Evaluate $\int \frac{(1-x)^2}{\sqrt[3]{x^2}} dx$

11. Evaluate $\int \frac{(1-x)^3}{x\sqrt[3]{x}} dx$

12. A particle starts from a point O and moves in a straight line with a velocity v ms^{-1}, given by $v = 25t - 6t^2$, where t seconds is the time after leaving O. Calculate

    (a) the initial ($t = 0$) acceleration of the particle

    (b) the value of t when the acceleration is zero

    (c) the value of t when the particle returns to O (i.e. displacement is zero)