Remedial Lesson 2: All the Differentiation You Needed

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1 Table of Differentiations

Rules	Formula
Addition Rule	$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$
Constant	$\frac{d}{dx}kf(x) = k\frac{d}{dx}f(x)$
Product Rule	$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$
Quotient Rule	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{g^2(x)}$
Chain Rule	$\frac{d}{dx}[f(g(x))] = \frac{d}{dg}f(g) \cdot \frac{d}{dx}g(x)$
Parametric Function	$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$
Inverse Function	$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1}$

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
k	0	$\sin x$	$\cos x$
x	1	$\cos x$	$-\sin x$
x^n	nx^{n-1}	$\tan x$	$\sec^2 x$
e^x	e^x	$\cot x$	$-\csc^2 x$
$\ln x$	$1/x$	$\sec x$	$\sec x \tan x$
		$\csc x$	$-\csc x \cot x$

2 Exponential Functions

- We have something called natural number, $e = 2.717828\ldots$

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

- Raising e to the power of x (i.e. $f(x) = e^x$) is called the exponential function. For convenience, we may write it as $f(x) = \exp(x)$
- Differentiation:

$$\frac{d}{dx}e^x = e^x$$

- Example:

$$\begin{aligned}\frac{d}{dx}e^{-\lambda x} =& \frac{d}{d(-\lambda x)}e^{-\lambda x}\frac{d}{dx}(-\lambda x)\\ =& e^{-\lambda x}\left(-\lambda\frac{d}{dx}x\right)\\ =& -\lambda e^{-\lambda x}\end{aligned}$$

3 Logarithmic Function

- If $y = e^x$, then we define $x = \ln y$. Where ln is the natural logarithm.
- Differentiation:

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

- Example:

$$\begin{aligned}\frac{d}{dx}\ln(x^2) =& \frac{d}{d(x^2)}\ln(x^2)\frac{d}{dx}(x^2)\\ =& \frac{1}{x^2}(2x)\\ =& \frac{2}{x}\end{aligned}$$

Exercises

1. Evaluate $\frac{d}{dx}\left[\ln(e^{2x}+e^{2a}) - \ln(e^{x-a}+e^{a-x}) + \frac{a}{x}\tan x\right]$

$$\begin{aligned}&\frac{d}{dx}\left[\ln(e^{2x}+e^{2a}) - \ln(e^{x-a}+e^{a-x}) + \frac{a}{x}\tan x\right]\\ =&\frac{d}{dx}\ln(e^{2x}+e^{2a}) - \frac{d}{dx}\ln(e^{x-a}+e^{a-x}) + \frac{d}{dx}\left(\frac{a}{x}\tan x\right)\\ =&\frac{1}{e^{2x}+e^{2a}}\left(e^{2x}(2)\right) - \frac{1}{e^{x-a}+e^{a-x}}\left(e^{x-a}(1)+e^{a-x}(-1)\right) + \left[\frac{a}{x}\sec^2 x + \tan x\frac{-a}{x^2}\right]\\ =&\frac{2e^{2x}}{e^{2x}+e^{2a}} - \frac{e^{x-a}-e^{a-x}}{e^{x-a}+e^{a-x}} + \frac{a}{x}\sec^2 x - \frac{a}{x^2}\tan x\\ =&\frac{2e^{2(x-a)}}{e^{2(x-a)}+1} - \frac{e^{2(x-a)}-1}{e^{2(x-a)}+1} + \frac{a}{x}\sec^2 x - \frac{a}{x^2}\tan x\\ =&\frac{e^{2(x-a)}+1}{e^{2(x-a)}+1} + \frac{a}{x}\sec^2 x - \frac{a}{x^2}\tan x\end{aligned}$$

2. Evaluate $\frac{d}{dx}(\ln\ln x)$

$$\begin{aligned}&\frac{d}{dx}(\ln\ln x)\\ =&\frac{1}{\ln x}\frac{d}{dx}(\ln x)\\ =&\frac{1}{x\ln x}\end{aligned}$$

3. Evaluate $\frac{d}{dx}\log_{10} x$

$$\begin{aligned}&\frac{d}{dx}\log_{10} x\\ =&\frac{d}{dx}\left(\frac{\ln x}{\ln 10}\right)\\ =&\frac{1}{\ln 10}\frac{d}{dx}\ln x\\ =&\frac{1}{x\ln 10}\end{aligned}$$

4. Evaluate $\frac{d}{dx}\exp(\tan x^2)$

$$\begin{aligned}&\frac{d}{dx}\exp(\tan x^2)\\ =&\exp(\tan x^2)\frac{d}{dx}\tan x^2\\ =&\exp(\tan x^2)\sec^2 x^2\frac{d}{dx}(x^2)\\ =&2xe^{\tan x^2}\sec^2 x^2\end{aligned}$$

5. If $f(x) = e^{-x/a}\cos(\frac{x}{a})$, find $f(0) + af'(0)$

$$\begin{aligned}\frac{d}{dx}f(x) =&\frac{d}{dx}\left[e^{-x/a}\cos\left(\frac{x}{a}\right)\right]\\ =&\cos\left(\frac{x}{a}\right)\frac{d}{dx}e^{-x/a} + e^{-x/a}\frac{d}{dx}\cos\left(\frac{x}{a}\right)\\ =&\cos\left(\frac{x}{a}\right)e^{-x/a}\frac{-1}{a} + e^{-x/a}\left(-\sin\left(\frac{x}{a}\right)\right)\left(\frac{1}{a}\right)\\ =&-\frac{1}{a}e^{-x/a}\cos\left(\frac{x}{a}\right) - \frac{1}{a}e^{-x/a}\sin\left(\frac{x}{a}\right)\\ =&-\frac{1}{a}e^{-x/a}\left[\cos\left(\frac{x}{a}\right)+\sin\left(\frac{x}{a}\right)\right]\\ \therefore\qquad f'(0) =&-\frac{1}{a}e^{0}\left[\cos(0)+\sin(0)\right]\\ =&-\frac{1}{a}\\ f(0) =&e^0\cos(0) = 1\\ \therefore\quad f(0)+af'(0) =&1+a\end{aligned}$$

6. Show that $y = \exp(2x)\sin x$ satisfies $y'' - 4y' + 5y = 0$

$$\begin{aligned}y =&e^{2x}\sin x\\ \therefore\qquad y' =&2e^{2x}\sin x + e^{2x}\cos x\\ y'' =&4e^{2x}\sin x + 2e^{2x}\cos x + 2e^{2x}\cos x - e^{2x}\sin x\\ =&3e^{2x}\sin x + 4e^{2x}\cos x\\ \therefore\quad y''-4y'+5y =&3e^{2x}\sin x + 4e^{2x}\cos x\\ &-8e^{2x}\sin x - 4e^{2x}\cos x\\ &+5e^{2x}\sin x\\ =&0\end{aligned}$$

4 Differentiation of Inverse Function

- Rule of Thumb:

$$\frac{dy}{dx} = \frac{1}{dx/dy}$$

4.1 Inverse Trigonometric Functions

- Example:

$$\begin{aligned}
\text{Let} \quad && x &= \sin y \\
\therefore \quad && y &= \sin^{-1} x \\
&& \frac{dx}{dy} &= \cos y \\
&& &= \sqrt{1-\sin^2 y} \\
&& &= \sqrt{1-x^2} \\
\therefore \quad && \frac{dy}{dx} &= \frac{1}{\sqrt{1-x^2}} \\
\therefore \quad && \frac{d}{dx}(\sin^{-1} x) &= \frac{1}{\sqrt{1-x^2}}
\end{aligned}$$

- Formulae to be used:

$$\begin{aligned}
\sin^2 x + \cos^2 x &= 1 \\
\sec^2 x &= \tan^2 x + 1 \\
\csc^2 x &= \cot^2 x + 1
\end{aligned}$$

- Complete the following table:

$f(x)$	$f'(x)$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\cot^{-1} x$	$-\frac{1}{1+x^2}$
$\sec^{-1} x$	$\frac{1}{x\sqrt{x^2-1}}$
$\csc^{-1} x$	$-\frac{1}{x\sqrt{x^2-1}}$

4.2 Other inverse functions

- Example:

$$\begin{aligned}
\text{Let} \quad y &= \sqrt{x} \\
\therefore \quad x &= y^2 \\
\frac{dx}{dy} &= 2y \\
&= 2\sqrt{x} \\
\frac{d}{dx}(\sqrt{x}) &= \frac{1}{2\sqrt{x}}
\end{aligned}$$

Exercises

1. If $y = \sin^{-1} \frac{2x-7}{3}$, find $\frac{dy}{dx}$

$$\begin{aligned}
y &= \sin^{-1} \frac{2x-7}{3} \\
\therefore \ \sin y &= \frac{2x-7}{3} \\
x &= \frac{3\sin y + 7}{2} \\
\frac{dx}{dy} &= \frac{3\cos y}{2} \\
&= \frac{3}{2}\sqrt{1 - \left(\frac{2x-7}{3}\right)^2} \\
&= \sqrt{-x^2 + 7x - 10} \\
&= \sqrt{(x-5)(2-x)} \\
\therefore \quad \frac{dy}{dx} &= \frac{1}{\sqrt{(x-5)(2-x)}}
\end{aligned}$$

2. If $y = \tan^{-1} e^x$, find $\frac{dy}{dx}$

$$\begin{aligned}
y &= \tan^{-1} e^x \\
\therefore \quad \frac{dy}{d(e^x)} &= \frac{1}{1 + (e^x)^2} \\
\therefore \quad \frac{dy}{dx} &= \frac{dy}{d(e^x)} \frac{d(e^x)}{dx} \\
&= \frac{e^x}{1 + (e^x)^2} \\
&= \frac{e^x}{1 + e^{2x}}
\end{aligned}$$

3. If $y = \sec^{-1} \tan x$, find $\frac{dy}{dx}$

$$\begin{aligned}
\frac{dy}{d(\tan x)} &= \frac{1}{\tan x \sqrt{\tan^2 x - 1}} \\
\therefore \quad \frac{dy}{dx} &= \frac{1}{\tan x \sqrt{\tan^2 x - 1}} \sec^2 x \\
&= \frac{\sec^2 x \cot x}{\sqrt{\tan^2 x - 1}} = \frac{\sec x \csc x}{\sqrt{\tan^2 x - 1}}
\end{aligned}$$

4. Show that if $y = (\sin^{-1} x)^2$, $(1-x^2)y'' - xy' = 2$

$$\begin{aligned}
y =& (\sin^{-1} x)^2 \\
\frac{dy}{dx} =& 2(\sin^{-1} x)\frac{1}{\sqrt{1-x^2}} \\
\frac{d^2y}{dx^2} =& 2\frac{1}{\sqrt{1-x^2}}\frac{1}{\sqrt{1-x^2}} + 2(\sin^{-1} x)\frac{-1}{2\sqrt{(1-x^2)^3}}(-2x) \\
=& 2\frac{1}{1-x^2} + \frac{2x(\sin^{-1} x)}{(1-x^2)^{3/2}} \\
\therefore \quad (1-x^2)y'' - xy' =& (1-x^2)\left(2\frac{1}{1-x^2} + \frac{2x(\sin^{-1} x)}{(1-x^2)^{3/2}}\right) \\
& - x\left(2(\sin^{-1} x)\frac{1}{\sqrt{1-x^2}}\right) \\
=& \left(2 + \frac{2x(\sin^{-1} x)}{(1-x^2)^{1/2}}\right) - \frac{2x(\sin^{-1} x)}{\sqrt{1-x^2}} \\
=& 2
\end{aligned}$$

5 Implicit Functions

- Implicit function: Those given as an equation, but not a function
- Example of __implicit__ function: Circle equation

$$(x-h)^2 + (y-k)^2 = r^2$$

- Example of __explicit__ function: Semicircle

$$y = k + \sqrt{r^2 - (x-h)^2}$$

- Differentiation of implicit function: Use the rules to differentiate both side, then simplify
- Example: Find $\frac{dy}{dx}$ from the standard form of circle equation

$$\begin{aligned}
(x-h)^2 + (y-k)^2 =& r^2 \\
\therefore \quad 2(x-h) + 2(y-k)\frac{dy}{dx} =& 0 \\
\frac{dy}{dx} =& -\frac{x-h}{y-k}
\end{aligned}$$

6 Application of Differentiation

6.1 Parametric functions

- Curves may be expressed as __parametric form__, such as circle, it can be expressed in standard form:

$$x^2 + y^2 = r^2$$

or in parametric form:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

- If a curve is presented in parametric form, the derivative $\frac{dy}{dx}$ is given by

$$\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta}$$

- Example: Use the parametric form of circle

$$\begin{cases} x = r\cos\theta + h \\ y = r\sin\theta + k \end{cases}$$

to find $\frac{dy}{dx}$.

$$\begin{aligned} x &= r\cos\theta + h \\ \therefore \quad \frac{dx}{d\theta} &= -r\sin\theta \\ y &= r\sin\theta + k \\ \therefore \quad \frac{dy}{d\theta} &= r\cos\theta \\ \therefore \quad \frac{dy}{dx} &= \frac{r\cos\theta}{-r\sin\theta} \\ &= \frac{x-h}{-(y-k)} \\ &= -\frac{x-h}{y-k} \end{aligned}$$

6.2 Find tangents and normals

- Tangents: Limit of chord on a curve
- Normals: Lines cutting the curve and perpendicular to the tangent at that point
- Differentation can help to find the slope, so that you can use straight line formulae to find the tangents or normals
- Example: Find the equation of tangent at point $(0,2)$ from the circle $x^2 + y^2 = 4$.

$$\begin{aligned} x^2 + y^2 &= 4 \\ \therefore \quad 2x + 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{x}{y} \\ \therefore \quad \left.\frac{dy}{dx}\right|_{(0,2)} &= \frac{0}{2} = 0 \\ \therefore \quad \text{Equation is: } y - 2 &= 0(x-0) \\ \implies \quad y &= 2 \end{aligned}$$

Exercises

1. Determine the constants A and B such that the normal to the curve $y = Ae^x + Be^{-x}$ at $(0,2)$ will be parallel to the line $3x - y = 4$

$$\begin{aligned}
y =& Ae^x + Be^{-x} \\
\frac{dy}{dx} =& Ae^x - Be^{-x} \\
\therefore \qquad \left.\frac{dy}{dx}\right|_{(0,2)} =& A - B = 3 \quad \text{(slope)} \\
2 =& Ae^0 + Be^0 = A + B \quad \text{(point)} \\
\implies \quad \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} =& \begin{bmatrix} 3 \\ 2 \end{bmatrix} \\
\begin{bmatrix} A \\ B \end{bmatrix} =& \begin{bmatrix} 5/2 \\ -1/2 \end{bmatrix}
\end{aligned}$$

2. Prove that curves $y = \exp(x^2)/ex$ and $y = x^2 - \ln x^3$ intersect at right angles at the point $(1,1)$

$$\begin{aligned}
y =& \frac{e^{x^2}}{ex} \\
\frac{dy}{dx} =& \frac{(ex)(e^{x^2})(2x) - (e^{x^2})(e)}{(ex)^2} \\
=& \frac{(e^{x^2})(2x^2 - 1)}{ex^2} \\
\left.\frac{dy}{dx}\right|_{(1,1)} =& \frac{(e^1)(2(1) - 1)}{e(1)} = 1 \quad \text{(slope)} \\
y =& x^2 - \ln x^3 \\
\frac{dy}{dx} =& 2x - \frac{1}{x^3}(3x^2) \\
=& 2x - \frac{3}{x} \\
\left.\frac{dy}{dx}\right|_{(1,1)} =& 2 - 3 = -1 \quad \text{(slope)}
\end{aligned}$$

$$(1)(-1) = -1 \qquad \implies \perp$$

3. Find the equation of the tangent at $t = t_1$ to the curve given by the parametric equations $\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}$

$$\begin{aligned}
\frac{dx}{dt} =& a(1 - \cos t) \\
\frac{dy}{dt} =& a \sin t \\
\therefore \qquad \frac{dy}{dx} =& \frac{\sin t}{1 - \cos t} \\
\left.\frac{dy}{dx}\right|_{t=t_1} =& \frac{\sin t_1}{1 - \cos t_1}
\end{aligned}$$

4. Find the equations of tangent and normal at the point $(4,3)$ to the curve given by the parametric equations $x = t^2$ and $y = 2t - 1$. Show that the normal cuts the curve again at the point where $t = -3$.

$$\begin{aligned}
\frac{dx}{dt} &= 2t \\
\frac{dy}{dt} &= 2 \\
\therefore \quad \frac{dy}{dx} &= \frac{2}{2t} = \frac{1}{t} \\
x = 4 \implies t &= \sqrt{4} = 2 \\
\therefore \quad \left.\frac{dy}{dx}\right|_{(4,3)} &= \frac{1}{2} \\
\text{Slope:} \quad y - 3 &= \frac{1}{2}(x-4) \\
\implies \quad x - 2y + 2 &= 0 \\
\text{Normal:} \quad y - 3 &= -2(x-4) \\
\implies \quad 2x + y - 11 &= 0 \\
\text{Sub } t = -3\text{:} \quad (x,y) &= (9,-7) \\
2(9) + (-7) - 11 &= 0 \\
\therefore \quad & (9,-7) \text{ on normal}
\end{aligned}$$

5. Find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $t = \frac{\pi}{2}$ in the following parametric equations: $\begin{cases} x = a(nt - \sin t) \\ y = a(t + \sin t) \end{cases}$

$$\begin{aligned}
\frac{dx}{dt} &= a(n - \cos t) \\
\frac{dy}{dt} &= a(1 + \cos t) \\
\therefore \quad \frac{dy}{dx} &= \frac{1 + \cos t}{n - \cos t} \\
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) \\
&= \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx} \\
&= \frac{(n - \cos t)(-\sin t) - (1 + \cos t)(\sin t)}{(n - \cos t)^2} \cdot \frac{1}{a(n - \cos t)} \\
&= \frac{-\sin t(n - 1 - 2\cos t)}{a(n - \cos t)^3} \\
\therefore \quad \left.\frac{dy}{dx}\right|_{t=\pi/2} &= \frac{1}{n} \\
\therefore \quad \left.\frac{d^2y}{dx^2}\right|_{t=\pi/2} &= \frac{-(n-1)}{a(n)^3} = \frac{1-n}{an^3}
\end{aligned}$$