Remedial Lesson 3: Indefinite Integrals

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1 Indefinite Integral as Inverse of Differentiation

- Rules:

Rules	Differentiation	Integration
Notation	$f'(x) = \frac{d}{dx} f(x)$	$\int f'(x) dx = f(x) + C$
Addition Rule	$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$	$\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$
Constant	$\frac{d}{dx} k f(x) = k \frac{d}{dx} f(x)$	$\int k f(x) dx = k \int f(x) dx$
Product Rule	$\frac{d}{dx}[f(x)g(x)] = f(x) \frac{d}{dx} g(x) + g(x) \frac{d}{dx} f(x)$	$\int f(x) g'(x) dx = f(x) g(x) - \int g(x) f'(x) dx$
Quotient Rule	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \frac{d}{dx} f(x) - f(x) \frac{d}{dx} g(x)}{g^2(x)}$	—
Chain Rule	$\frac{d}{dx}[f(g(x))] = \frac{d}{dg} f(g) \cdot \frac{d}{dx} g(x)$	$\int f(g(x)) dx = \int f'(g) g'(x) dg$
Parametric Function	$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$	—
Inverse Function	$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1}$	—

- Table of Differentiation:

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
k	0	$\sin kx$	$k \cos kx$	$\sin^{-1} kx$	$\frac{k}{\sqrt{1-(kx)^2}}$
x	1	$\cos kx$	$-k \sin kx$	$\cos^{-1} kx$	$-\frac{k}{\sqrt{1-(kx)^2}}$
x^n	nx^{n-1}	$\tan kx$	$k \sec^2 kx$	$\tan^{-1} kx$	$\frac{k}{1+(kx)^2}$
a^{kx}	$(k \ln a) a^{kx}$	$\cot kx$	$-k \csc^2 kx$	$\cot^{-1} kx$	$-\frac{k}{1+(kx)^2}$
e^{kx}	ke^{kx}	$\sec kx$	$k \sec kx \tan kx$	$\sec^{-1} kx$	$\frac{1}{x\sqrt{(kx)^2-1}}$
$\ln(kx)$	$\frac{k}{x}$	$\csc kx$	$-k \csc kx \cot kx$	$\csc^{-1} kx$	$-\frac{k}{x\sqrt{(kx)^2-1}}$

Exercises

1. Find $\int dx$

$$\begin{aligned} &\int dx \\ =&\int (1)dx \\ =&x+C \end{aligned}$$

2. Find $\int x^n dx$

$$\begin{aligned} &\int x^n dx \\ =&\frac{1}{n+1}\int (n+1)x^n dx \\ \therefore \quad \int x^n dx =&x^{n+1}+C \end{aligned}$$

3. Find $\int \frac{1}{x}dx$, where $x>0$

$$\begin{aligned} &\int \frac{dx}{x} \\ =&\ln x+C \end{aligned}$$

4. Find $\int e^{-\lambda x}dx$

$$\begin{aligned} &\int e^{-\lambda x}dx \\ =&-\lambda e^{-\lambda x}+C \end{aligned}$$

5. Find $\int(\int(\int(\int \sin x dx)dx)dx)dx$

$$\begin{aligned} &\int(\int(\int(\int \sin x dx)dx)dx)dx \\ =&\int(\int(\int(-\cos x+C_1)dx)dx)dx \\ =&\int(\int(-\sin x+C_1x+C_2)dx)dx \\ =&\int(\cos x+\frac{C_1}{2}x^2+C_2x+C_3)dx \\ =&\sin x+\frac{C_1}{6}x^3+\frac{C_2}{2}x^2+C_3x+C_4 \end{aligned}$$

6. Find $\underbrace{\int\int\cdots\int}_{n}(1)\underbrace{dxdx\cdots dx}_{n}$

$$\begin{aligned} &\int(1)dx \\ =&x+C \end{aligned}$$

$$\begin{aligned} &\iint(1)dxdx \\ =&\int(x+C_1)dx \\ =&\frac{1}{2}x^2+C_1x+C_2 \end{aligned}$$

$$
\begin{aligned}
&\iiint (1)dxdxdx \\
=&\int (\frac{1}{2}x^2+C_1x+C_2)dx \\
=&\frac{1}{2(3)}x^3+\frac{C_1}{2}x^2+C_2x+C_3
\end{aligned}
$$

$$
\begin{aligned}
\therefore \quad &\iint\cdots\int (1)dxdx\cdots dx \\
=&\frac{1}{n!}x^n+\sum_{k=0}^{n-1}\frac{C_k}{k!}x^k
\end{aligned}
$$

2 Differentials, and Integration by Change of Variable

- *Think* of derivatives as if they are fractions

$$\frac{dy}{dx} = dy \div dx$$

- Then we have:

$$
\begin{aligned}
y &= f(x) \\
\frac{dy}{dx} &= f'(x) \\
dy &= f'(x)dx
\end{aligned}
$$

which we call dy as the differential of y

- So, we can have the method of substitution for solving indefinite integrals:

$$
\begin{aligned}
&\int g'(y)f'(x)dx \\
=&\int g'(y)dy \\
=&g(y)+C
\end{aligned}
$$

- Example:

$$
\begin{aligned}
&\int \frac{2x+3}{(x^2+3x+2)^6}dx \\
=&\int \frac{1}{(x^2+3x+2)^6}\frac{d}{dx}(x^2+3x+2)dx \\
=&\int \frac{1}{(x^2+3x+2)^6}d(x^2+3x+2) \\
=&\int (x^2+3x+2)^{-6}d(x^2+3x+2) \\
=&\frac{1}{-5}(x^2+3x+2)^{-5}+C
\end{aligned}
$$

$$
\begin{aligned}
&\int \frac{2x+3}{(x^2+3x+2)^6}dx && \text{(sub } y=x^2+3x+2\text{)} \\
=&\int \frac{y'}{y^6}dx \\
=&\int \frac{1}{y^6}dy && \text{(note: } dy=y'dx\text{)} \\
=&\int y^{-6}dy \\
=&\frac{1}{-5}y^{-5}+C \\
=&\frac{1}{-5}(x^2+3x+2)^{-5}+C
\end{aligned}
$$

- Example: Find $\int \frac{1}{x}dx$ where $x<0$

$$\begin{aligned}
&\int \frac{1}{x}dx && (\text{sub } y=-x)\\
=&-\int \frac{1}{y}dx\\
=&\int \frac{1}{y}dy && (dy=-dx)\\
=&\ln y+C\\
=&\ln(-x)+C
\end{aligned}$$

Note: Therefore, we have a formula:

$$\int \frac{dx}{x}=\ln|x|+C$$

- Example again:

$$\begin{aligned}
&\int \tan x dx\\
=&\int \frac{\sin x}{\cos x}dx\\
=&\int \frac{-1}{\cos x}d(\cos x)\\
=&-\int \frac{d(\cos x)}{\cos x}\\
=&-\ln|\cos x|+C\\
=&\ln|\sec x|+C
\end{aligned}$$

$$\begin{aligned}
&\int \tan x dx\\
=&\int \frac{\sin x}{\cos x}dx && (\text{sub } y=\cos x)\\
=&\int \frac{-1}{y}dy && (dy=-\sin x dx)\\
=&-\int \frac{dy}{y}\\
=&-\ln|y|+C\\
=&\ln\left|\frac{1}{y}\right|+C\\
=&\ln\left|\frac{1}{\cos x}\right|+C\\
=&\ln|\sec x|+C
\end{aligned}$$

Exercises

1. Find $\int \cot x dx$

$$\begin{aligned}
&\int \cot x dx=\int \frac{\cos x}{\sin x}dx\\
=&\int \frac{1}{\sin x}d(\sin x)\\
=&\ln|\sin x|+C
\end{aligned}$$

2. Find $\int \sec x dx$ (Hint: Substitute $u=\sec x+\tan x$)

$$\begin{aligned}
&\int \sec x dx\\
&=\int \frac{\sec x(\sec x+\tan x)}{\sec x+\tan x}dx=\int \frac{\sec^2 x+\sec x\tan x}{\sec x+\tan x}dx\\
du&=(\sec x\tan x+\sec^2 x)dx\\
\therefore \quad \int \sec x dx&=\int \frac{d(\sec x+\tan x)}{\sec x+\tan x}=\int \frac{du}{u}\\
&=\ln|u|+C\\
&=\ln|\sec x+\tan x|+C
\end{aligned}$$

3. Find $\int \csc x dx$ (Hint: Substitute $u = \csc x - \cot x$)

$$\begin{aligned}
&\int \csc x dx \\
&= \int \frac{\csc x(\csc x - \cot x)}{\csc x - \cot x} dx \\
&= \int \frac{\csc^2 x - \csc x \cot x}{\csc x - \cot x} dx \\
du &= -\csc x \cot x + \csc^2 x \\
\therefore \quad \int \csc x dx &= \int \frac{d(\csc x - \cot x)}{\csc x - \cot x} = \int \frac{du}{u} \\
&= \ln|u| + C \\
&= \ln|\csc x - \cot x| + C
\end{aligned}$$

4. Find $\int x^2 \exp(x^3) dx$

$$\begin{aligned}
&\int x^2 e^{(x^3)} dx \\
=&\frac{1}{3}\int e^{(x^3)} d(x^3) \\
=&\frac{1}{3} e^{(x^3)} + C
\end{aligned}$$

3 Substitution of Trigonometric Functions

- Useful identites of trigonometry:

$$\begin{array}{lll}
\sin^2 x + \cos^2 x = 1 & \sin x = \sqrt{1-\cos^2 x} & \cos x = \sqrt{1-\sin^2 x} \\
\sec^2 x - \tan^2 x = 1 & \sec x = \sqrt{1+\tan^2 x} & \tan x = \sqrt{\sec^2 x - 1} \\
\csc^2 x - \cot^2 x = 1 & \csc x = \sqrt{1+\cot^2 x} & \cot x = \sqrt{\csc^2 x - 1}
\end{array}$$

- Trigonometric laws:

$$\begin{aligned}
\sin(A+B) &= \sin A \cos B + \cos A \sin B & \sin 2A &= 2\sin A \cos B \\
\sin(A-B) &= \sin A \cos B - \cos A \sin B & \cos 2A &= \cos^2 A - \sin^2 A \\
\cos(A+B) &= \cos A \cos B + \sin A \sin B & &= 1 - 2\sin^2 A \\
\cos(A-B) &= \cos A \cos B - \sin A \sin B & &= 2\cos^2 A - 1 \\
\tan(A+B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B} & \tan 2A &= \frac{2\tan A}{1-\tan^2 A} \\
\tan(A-B) &= \frac{\tan A - \tan B}{1 + \tan A \tan B}
\end{aligned}$$

$$\begin{array}{lll}
\sin^2 x = \dfrac{1-\cos 2x}{2} & \sin\dfrac{x}{2} = \pm\sqrt{\dfrac{1-\cos x}{2}} & \sin x = \dfrac{2t}{1+t^2} \\
\cos^2 x = \dfrac{1+\cos 2x}{2} & \cos\dfrac{x}{2} = \pm\sqrt{\dfrac{1+\cos x}{2}} & \cos x = \dfrac{1-t^2}{1+t^2} \\
\tan^2 x = \dfrac{1-\cos 2x}{1+\cos 2x} & \tan\dfrac{x}{2} = \pm\sqrt{\dfrac{1-\cos x}{1+\cos x}} & \tan x = \dfrac{2t}{1-t^2} \\
 & \quad = \dfrac{\sin x}{1+\cos x} & (t = \tan\dfrac{x}{2}) \\
 & \quad = \dfrac{\cos x}{1+\sin x} &
\end{array}$$

- Sum-to-product formulae:

$$\sin x+\sin y=2\sin\frac{x+y}{2}\cos\frac{x-y}{2} \qquad \sin x\sin y=-\frac{1}{2}[\cos(x+y)-\cos(x-y)]$$
$$\sin x-\sin y=2\cos\frac{x+y}{2}\sin\frac{x-y}{2} \qquad \cos x\cos y=\frac{1}{2}[\cos(x+y)+\cos(x-y)]$$
$$\cos x+\cos y=2\cos\frac{x+y}{2}\cos\frac{x-y}{2} \qquad \sin x\cos y=\frac{1}{2}[\sin(x+y)+\sin(x-y)]$$
$$\cos x-\cos y=-2\sin\frac{x+y}{2}\sin\frac{x-y}{2} \qquad \cos x\sin y=\frac{1}{2}[\sin(x+y)-\sin(x-y)]$$

- Substitution using trigonometric functions: (Example)

$$\begin{aligned}
&\int\sqrt{1-x^2}dx && (\text{sub } x=\cos t)\\
=&\int\sin t\,d(\cos t)\\
=&-\int\sin^2 t\,dt\\
=&-\int\frac{1-\cos 2t}{2}dt\\
=&-\frac{1}{2}t+\frac{1}{2}\int\cos(2t)\frac{d(2t)}{2}\\
=&-\frac{t}{2}+\frac{1}{4}\sin(2t)+C\\
=&-\frac{t}{2}+\frac{2\sin t\cos t}{4}+C\\
=&-\frac{1}{2}\cos^{-1}x+\frac{1}{2}x\sqrt{1-x^2}+C\\
=&\frac{1}{2}\left(x\sqrt{1-x^2}-\cos^{-1}x\right)+C
\end{aligned}$$

- Another example:

$$\begin{aligned}
&\int\frac{dx}{\sqrt{x^2+4}} && (\text{sub } x=2\tan t)\\
=&\int\frac{d(2\tan t)}{\sqrt{4\tan^2 t+4}}\\
=&\int\frac{2\sec^2 t\,dt}{2\sec t}\\
=&\int\sec t\,dt\\
=&\ln|\sec t+\tan t|+C\\
=&\ln\left|\sqrt{\left(\frac{2\tan t}{2}\right)^2+1}+\frac{2\tan t}{2}\right|+C\\
=&\ln\left|\sqrt{\frac{x^2}{4}+1}+\frac{x}{2}\right|+C\\
=&\ln\left|x+\sqrt{x^2+4}\right|-\ln 2+C\\
=&\ln\left|x+\sqrt{x^2+4}\right|+C'
\end{aligned}$$

- Summary:

1. When you see $\sqrt{a^2-x^2}$, use $x=a\sin t$ or $x=a\cos t$
2. When you see $\sqrt{a^2+x^2}$, use $x=a\tan t$ or $x=a\cot t$
3. When you see $\sqrt{x^2-a^2}$, use $x=a\sec t$ or $x=a\csc t$

Exercises

1. Find $\int \frac{dx}{\sqrt{a^2-x^2}}$

$$\begin{aligned}
&\int \frac{dx}{\sqrt{a^2-x^2}} \\
=&\int \frac{a\cos t\,dt}{\sqrt{a^2-a^2\sin^2 t}} && (\text{sub } x = a\sin t) \\
=&\int dt \\
=&t+C \\
=&\sin^{-1}\left(\frac{x}{a}\right)+C
\end{aligned}$$

2. Find $\int \frac{dx}{\sqrt{x^2-a^2}}$

$$\begin{aligned}
&\int \frac{dx}{\sqrt{x^2-a^2}} \\
=&\int \frac{a\sec t\tan t\,dt}{\sqrt{a^2\sec^2 t-a^2}} && (\text{sub } x = a\sec t) \\
=&\int \sec t\,dt \\
=&\ln|\sec t+\tan t|+C \\
=&\ln\left|\frac{x}{a}+\sqrt{\frac{x^2}{a^2}-1}\right|+C \\
=&\ln\left|x+\sqrt{x^2-a^2}\right|-\ln|a|+C \\
=&\ln\left|x+\sqrt{x^2-a^2}\right|+C'
\end{aligned}$$

3. Find $\int \frac{dx}{\sqrt{a^2+x^2}}$

$$\begin{aligned}
&\int \frac{dx}{\sqrt{a^2+x^2}} \\
=&\int \frac{a\sec^2 t\,dt}{\sqrt{a^2+a^2\tan^2 t}} && (\text{sub } x = a\tan t) \\
=&\int \sec t\,dt \\
=&\ln|\sec t+\tan t|+C \\
=&\ln\left|\sqrt{\frac{x^2}{a^2}+1}+\frac{x}{a}\right|+C \\
=&\ln\left|x+\sqrt{x^2+a^2}\right|-\ln|a|+C \\
=&\ln\left|x+\sqrt{x^2+a^2}\right|C'
\end{aligned}$$

4. Find $\int \sqrt{a^2-x^2}dx$

$$
\begin{aligned}
&\int \sqrt{a^2-x^2}dx \\
=&\int a\cos t\sqrt{a^2-a^2\sin^2 t}dt && \text{(sub } x=a\sin t\text{)}\\
=&a^2\int \cos^2 t dt \\
=&a^2\int \frac{1+\cos 2t}{2}dt \\
=&\frac{a^2}{2}\left[t+\int \cos 2t dt\right] \\
=&\frac{a^2 t}{2}+\frac{a^2}{4}\int \cos 2t d(2t) && \text{(or sub } y=2t\text{)}\\
=&\frac{a^2 t}{2}+\frac{a^2}{4}\sin 2t+C \\
=&\frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right)+\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right]+C \\
=&\frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right)+\frac{x}{a^2}\sqrt{a^2-x^2}\right]+C \\
=&\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+\frac{x}{2}\sqrt{a^2-x^2}+C
\end{aligned}
$$

5. Find $\int \sqrt{x^2-a^2}dx$ (Cannot complete without further knowledge of integration)

6. Find $\int \sqrt{a^2+x^2}dx$ (Cannot complete without further knowledge of integration)

7. Find $\int \frac{1}{\sqrt{1+x+x^2}}dx$

$$
\begin{aligned}
&\int \frac{1}{\sqrt{1+x+x^2}}dx \\
=&\int \frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}} \\
=&\int \frac{d(x+\frac{1}{2})}{\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}} \\
=&\ln\left|(x+\frac{1}{2})+\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right|+C \\
=&\ln\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right|+C
\end{aligned}
$$

4 Bring-Home Exercises

1. $\int \frac{dx}{x(1+\ln x)^2}$

2. $\int e^{\sin x}\cos x dx$

3. $\int \frac{dx}{x\ln x}$

4. $\int (e^x+1)^2 dx$

5. $\int \frac{dx}{x^2\sqrt{x^2+a^2}}$

6. $\int \frac{dx}{x^4\sqrt{2+x^2}}$

7. $\displaystyle\int \frac{\sqrt{x^2-a^2}}{x^4}\,dx$

8. $\displaystyle\int \frac{dx}{(x^2+2x+3)^{3/2}}$

9. $\displaystyle\int \frac{dx}{x\sqrt{8x^2+2x-1}}$