Remedial Lesson 3: Indefinite Integrals

Prepared by Adrian Sai-wah TAM (swtam3@ie.cuhk.edu.hk)

September 26, 2005

1 Indefinite Integral as Inverse of Differentiation

- Rules:

Rules	Differentiation	Integration
Notation	$f'(x) = \frac{d}{dx} f(x)$	$\int f'(x)dx = f(x) + C$
Addition Rule		$\int [f(x) \pm g(x)]\, dx = \int f(x)dx \pm \int g(x)dx$
Constant		$\int k f(x)dx = k \int f(x)dx$
Product Rule	$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$	$\int f(x)g'(x)dx = f(x)g(x) - \int \qquad dx$
Quotient Rule		—
Chain Rule	$\frac{d}{dx}[f(g(x))] = \frac{d}{dg}f(g) \cdot \frac{d}{dx}g(x)$	$\int f(g(x))dx = \int f'(g)g'(x)dg$
Parametric Function	$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$	—
Inverse Function	$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1}$	—

- Table of Differentiation:

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
k	0	$\sin kx$	$k\cos kx$	$\sin^{-1} kx$	$\frac{k}{\sqrt{1-(kx)^2}}$
x	1	$\cos kx$	$-k\sin kx$	$\cos^{-1} kx$	$-\frac{k}{\sqrt{1-(kx)^2}}$
x^n	nx^{n-1}	$\tan kx$	$k\sec^2 kx$	$\tan^{-1} kx$	$\frac{k}{1+(kx)^2}$
a^{kx}	$(k\ln a)a^{kx}$	$\cot kx$	$-k\csc^2 kx$	$\cot^{-1} kx$	$-\frac{k}{1+(kx)^2}$
e^{kx}	ke^{kx}	$\sec kx$	$k\sec kx\tan kx$	$\sec^{-1} kx$	$\frac{1}{x\sqrt{(kx)^2-1}}$
$\ln(kx)$	$\frac{k}{x}$	$\csc kx$	$-k\csc kx\cot kx$	$\csc^{-1} kx$	$-\frac{k}{x\sqrt{(kx)^2-1}}$

Exercises

1. Find $\int dx$

$$\begin{aligned} &\int dx \\ =&\int (1) dx \\ =&x+C \end{aligned}$$

2. Find $\int x^n dx$

3. Find $\int \frac{1}{x} dx$, where $x > 0$

$$\begin{aligned} &\int \frac{dx}{x} \\ = \end{aligned}$$

4. Find $\int e^{-\lambda x} dx$

$$\begin{aligned} &\int e^{-\lambda x} dx \\ =& \qquad\qquad +C \end{aligned}$$

5. Find $\int (\int (\int (\int \sin x dx) dx) dx) dx$

$$\begin{aligned} &\int (\int (\int (\int \sin x dx) dx) dx) dx \\ =&\int (\int (\int (-\cos x + C_1) dx) dx) dx \\ =&\int (\int (-\sin x + C_1 x + C_2) dx) dx \\ =&\int (\cos x + \frac{C_1}{2} x^2 + C_2 x + C_3) dx \\ =&\sin x + \frac{C_1}{6} x^3 + \frac{C_2}{2} x^2 + C_3 x + C_4 \end{aligned}$$

6. Find $\underbrace{\int\int \cdots \int}_{n} (1) \underbrace{dx dx \cdots dx}_{n}$

$$\begin{aligned} &\int (1) dx \\ = \end{aligned}$$

$$\begin{aligned} &\iint (1) dx dx \\ =&\int (x + C_1) dx \\ = \end{aligned}$$

$$\begin{aligned}
&\iiint (1)dxdxdx \\
=&\int (\frac{1}{2}x^2 + C_1 x + C_2)dx \\
=&
\end{aligned}$$

$$\begin{aligned}
\therefore \quad &\iint \cdots \int (1)dxdx\cdots dx \\
=&
\end{aligned}$$

2 Differentials, and Integration by Change of Variable

- *Think* of derivatives as if they are fractions

$$\frac{dy}{dx} = dy \div dx$$

- Then we have:

$$\begin{aligned}
y &= f(x) \\
\frac{dy}{dx} &= f'(x) \\
dy &= f'(x)dx
\end{aligned}$$

  which we call dy as the differential of y

- So, we can have the method of substitution for solving indefinite integrals:

$$\begin{aligned}
&\int g'(y)f'(x)dx \\
=&\int g'(y)dy \\
=&g(y) + C
\end{aligned}$$

- Example:

$$\begin{aligned}
&\int \frac{2x+3}{(x^2+3x+2)^6}dx \\
=&\int \frac{1}{(x^2+3x+2)^6}\frac{d}{dx}(x^2+3x+2)dx \\
=&\int \frac{1}{(x^2+3x+2)^6}d(x^2+3x+2) \\
=&\int (x^2+3x+2)^{-6}d(x^2+3x+2) \\
=&\frac{1}{-5}(x^2+3x+2)^{-5}+C
\end{aligned}$$

$$\begin{aligned}
&\int \frac{2x+3}{(x^2+3x+2)^6}dx && \text{(sub } y = x^2+3x+2\text{)} \\
=&\int \frac{y'}{y^6}dx \\
=&\int \frac{1}{y^6}dy && \text{(note: } dy = y'dx\text{)} \\
=&\int y^{-6}dy \\
=&\frac{1}{-5}y^{-5}+C \\
=&\frac{1}{-5}(x^2+3x+2)^{-5}+C
\end{aligned}$$

- Example: Find $\int \frac{1}{x}dx$ where $x < 0$

$$\begin{aligned} & \int \frac{1}{x}dx && (\text{sub } y = -x) \\ = & -\int \frac{1}{y}dx \\ = & \int \frac{1}{y}dy && (dy = -dx) \\ = & \ln y + C \\ = & \end{aligned}$$

Note: Therefore, we have a formula:

$$\int \frac{dx}{x} = \ln|x| + C$$

- Example again:

$$\begin{aligned} & \int \tan x dx \\ = & \int \frac{\sin x}{\cos x}dx \\ = & \int \frac{-1}{\cos x}d(\cos x) \\ = & -\int \frac{d(\cos x)}{\cos x} \\ = & -\ln|\cos x| + C \\ = & \ln|\sec x| + C \end{aligned}$$

$$\begin{aligned} & \int \tan x dx \\ = & \int \frac{\sin x}{\cos x}dx && (\text{sub } y = \cos x) \\ = & \int \frac{-1}{y}dy && (dy = -\sin x dx) \\ = & -\int \frac{dy}{y} \\ = & -\ln|y| + C \\ = & \ln\left|\frac{1}{y}\right| + C \\ = & \ln\left|\frac{1}{\cos x}\right| + C \\ = & \ln|\sec x| + C \end{aligned}$$

Exercises

1. Find $\int \cot x dx$

2. Find $\int \sec x dx$ (Hint: Substitute $u = \sec x + \tan x$)

3. Find $\int \csc x dx$ (Hint: Substitute $u = \csc x - \cot x$)

4. Find $\int x^2 \exp(x^3) dx$

3 Substitution of Trigonometric Functions

- Useful identites of trigonometry:

$$\sin^2 x + \cos^2 x = 1 \qquad \sin x = \sqrt{1-\cos^2 x} \qquad \cos x = \sqrt{1-\sin^2 x}$$
$$\sec^2 x - \tan^2 x = 1 \qquad \sec x = \sqrt{1+\tan^2 x} \qquad \tan x = \sqrt{\sec^2 x - 1}$$
$$\csc^2 x - \cot^2 x = 1 \qquad \csc x = \sqrt{1+\cot^2 x} \qquad \cot x = \sqrt{\csc^2 x - 1}$$

- Trigonometric laws:

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$
$$\cos(A+B) = \cos A\cos B + \sin A\sin B$$
$$\cos(A-B) = \cos A\cos B - \sin A\sin B$$
$$\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A\tan B}$$
$$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B}$$

$$\sin 2A = 2\sin A\cos B$$
$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$$
$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2} \qquad \tan^2 x = \frac{1-\cos 2x}{1+\cos 2x}$$

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}} \qquad \cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$
$$\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{\sin x}{1+\cos x} = \frac{\cos x}{1+\sin x}$$

$$\sin x = \frac{2t}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2} \qquad \tan x = \frac{2t}{1-t^2} \qquad (t = \tan\frac{x}{2})$$

- Sum-to-product formulae:

$$\sin x+\sin y=2\sin\frac{x+y}{2}\cos\frac{x-y}{2} \qquad \sin x\sin y=-\frac{1}{2}\left[\cos(x+y)-\cos(x-y)\right]$$
$$\sin x-\sin y=2\cos\frac{x+y}{2}\sin\frac{x-y}{2} \qquad \cos x\cos y=\frac{1}{2}\left[\cos(x+y)+\cos(x-y)\right]$$
$$\cos x+\cos y=2\cos\frac{x+y}{2}\cos\frac{x-y}{2} \qquad \sin x\cos y=\frac{1}{2}\left[\sin(x+y)+\sin(x-y)\right]$$
$$\cos x-\cos y=-2\sin\frac{x+y}{2}\sin\frac{x-y}{2} \qquad \cos x\sin y=\frac{1}{2}\left[\sin(x+y)-\sin(x-y)\right]$$

- Substitution using trigonometric functions: (Example)

$$\begin{aligned}
&\int\sqrt{1-x^2}dx && \text{(sub } x=\cos t\text{)}\\
=&\int\sin t\,d(\cos t)\\
=&-\int\sin^2 t\,dt\\
=&-\int\frac{1-\cos 2t}{2}dt\\
=&-\frac{1}{2}t+\frac{1}{2}\int\cos(2t)\frac{d(2t)}{2}\\
=&-\frac{t}{2}+\frac{1}{4}\sin(2t)+C\\
=&-\frac{t}{2}+\frac{2\sin t\cos t}{4}+C\\
=&-\frac{1}{2}\cos^{-1}x+\frac{1}{2}x\sqrt{1-x^2}+C\\
=&\frac{1}{2}\left(x\sqrt{1-x^2}-\cos^{-1}x\right)+C
\end{aligned}$$

- Another example:

$$\begin{aligned}
&\int\frac{dx}{\sqrt{x^2+4}} && \text{(sub } x=2\tan t\text{)}\\
=&\int\frac{d(2\tan t)}{\sqrt{4\tan^2 t+4}}\\
=&\int\frac{2\sec^2 t\,dt}{2\sec t}\\
=&\int\sec t\,dt\\
=&\ln|\sec t+\tan t|+C\\
=&\ln\left|\sqrt{\left(\frac{2\tan t}{2}\right)^2+1}+\frac{2\tan t}{2}\right|+C\\
=&\ln\left|\sqrt{\frac{x^2}{4}+1}+\frac{x}{2}\right|+C\\
=&\ln\left|x+\sqrt{x^2+4}\right|-\ln 2+C\\
=&\ln\left|x+\sqrt{x^2+4}\right|+C'
\end{aligned}$$

- Summary:
  1. When you see $\sqrt{a^2-x^2}$, use $x=a\sin t$ or $x=a\cos t$
  2. When you see $\sqrt{a^2+x^2}$, use $x=a\tan t$ or $x=a\cot t$
  3. When you see $\sqrt{x^2-a^2}$, use $x=a\sec t$ or $x=a\csc t$

Exercises

1. Find $\int \frac{dx}{\sqrt{a^2 - x^2}}$

2. Find $\int \frac{dx}{\sqrt{x^2 - a^2}}$

3. Find $\int \frac{dx}{\sqrt{a^2 + x^2}}$

4. Find $\int \sqrt{a^2 - x^2}dx$

5. Find $\int \sqrt{x^2 - a^2} dx$ (Cannot complete without further knowledge of integration)

6. Find $\int \sqrt{a^2 + x^2} dx$ (Cannot complete without further knowledge of integration)

7. Find $\int \frac{1}{\sqrt{1+x+x^2}} dx$

4 Bring-Home Exercises

1. $\int \frac{dx}{x(1+\ln x)^2}$

2. $\int e^{\sin x} \cos x dx$

3. $\int \frac{dx}{x \ln x}$

4. $\int (e^x + 1)^2 dx$

5. $\int \frac{dx}{x^2 \sqrt{x^2 + a^2}}$

6. $\int \frac{dx}{x^4 \sqrt{2 + x^2}}$

7. $\int \frac{\sqrt{x^2 - a^2}}{x^4} dx$

8. $\int \frac{dx}{(x^2 + 2x + 3)^{3/2}}$

9. $\int \frac{dx}{x\sqrt{8x^2 + 2x - 1}}$