

Remedial Lesson 4: More Indefinite Integrals

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Formulas for Integration

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
1	x	$\sin kx$	$-\frac{1}{k} \cos kx$	$\frac{1}{\sqrt{a^2 - x^2}}$	$\sin^{-1} \frac{x}{a}$
x^n	$\frac{x^{n+1}}{n+1}$	$\cos kx$	$\frac{1}{k} \sin kx$	$\frac{1}{\sqrt{x^2 - a^2}}$	$\ln \left x + \sqrt{x^2 - a^2} \right $
a^{kx}	$\frac{a^{kx}}{(k \ln a)}$	$\tan x$	$\ln \sec x $	$\frac{1}{\sqrt{x^2 + a^2}}$	$\ln \left x + \sqrt{x^2 + a^2} \right $
e^{kx}	$\frac{1}{k} e^{kx}$	$\cot x$	$\ln \sin x $	$\frac{1}{x^2 - a^2}$	$\frac{1}{2a} \ln \left \frac{x-a}{x+a} \right $
$\frac{1}{x}$	$\ln x$	$\sec x$	$\ln \sec x + \tan x $	$\frac{1}{x^2 + a^2}$	$\frac{1}{a} \tan^{-1} \frac{x}{a}$
		$\csc x$	$\ln \csc x - \cot x $	$\sqrt{a^2 - x^2}$	$\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$
		$\sec kx \tan kx$	$\frac{1}{k} \sec kx$	$\sqrt{x^2 - a^2}$	$\frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln \left x + \sqrt{x^2 - a^2} \right $
		$\csc kx \cot kx$	$-\frac{1}{k} \csc kx$	$\sqrt{x^2 + a^2}$	$\frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \ln \left x + \sqrt{x^2 + a^2} \right $
		$\sec^2 kx$	$\frac{1}{k} \tan kx$		
		$\csc^2 kx$	$-\frac{1}{k} \cot kx$		

1 Integration by Part

- This is the product rule:

$$\begin{aligned} \frac{d}{dx} f(x)g(x) &= f(x) \frac{dg}{dx} + g(x) \frac{df}{dx} \\ \int \frac{d}{dx} f(x)g(x) dx &= \int f(x) \frac{dg}{dx} dx + \int g(x) \frac{df}{dx} dx \\ f(x)g(x) &= \int f(x) \frac{dg}{dx} dx + \int g(x) \frac{df}{dx} dx \\ \int f(x)g'(x) dx &= f(x)g(x) - \int g(x)f'(x) dx \end{aligned}$$

which has the form:

$$\int uv' dx = uv - \int vu' dx$$

or we memorize this as:

$$\int u dv = uv - \int v du$$

- Example of use:

$$\begin{aligned}
 & \int \ln x dx \\
 &= x \ln x - \int x d(\ln x) \\
 &= x \ln x - \int x \cdot \frac{1}{x} dx \\
 &= x \ln x - \int (1) dx \\
 &= x \ln x - x + C
 \end{aligned}$$

Examples

1. Evaluate $\int x^3 \ln x dx$

$$\begin{aligned}
 & \int x^3 \ln x dx \\
 &= \int \ln x d\left(\frac{1}{4}x^4\right) \\
 &= \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^4 d(\ln x) \\
 &= \frac{1}{4}x^4 \ln x - \frac{1}{4} \int x^3 dx \\
 &= \frac{1}{4}x^4 \ln x - \frac{1}{4} \left(\frac{1}{4}x^4\right) + C \\
 &= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C
 \end{aligned}$$

2. Evaluate $\int x^2 e^x dx$

$$\begin{aligned}
 & \int x^2 e^x dx \\
 &= \int x^2 d(e^x) \\
 &= x^2 e^x - \int e^x d(x^2) \\
 &= x^2 e^x - 2 \int x e^x dx \\
 &= x^2 e^x - 2 \int x d(e^x) \\
 &= x^2 e^x - 2 \left[x e^x - \int e^x dx \right] \\
 &= x^2 e^x - 2x e^x + 2e^x + C \\
 &= e^x(x^2 - 2x + 2) + C
 \end{aligned}$$

3. Evaluate $\int \frac{x \exp(x)}{(1+x)^2} dx$

$$\begin{aligned}
 & \int \frac{x e^x}{(1+x)^2} dx \\
 &= - \int x e^x d\left(\frac{1}{1+x}\right) \\
 &= - \frac{x e^x}{1+x} + \int \frac{1}{1+x} d(x e^x) \\
 &= - \frac{x e^x}{1+x} + \int \frac{e^x + x e^x}{1+x} dx
 \end{aligned}$$

$$\begin{aligned}
&= -\frac{xe^x}{1+x} + \int \frac{e^x(1+x)}{1+x} dx \\
&= -\frac{xe^x}{1+x} + \int e^x dx \\
&= -\frac{xe^x}{1+x} + e^x + C
\end{aligned}$$

Exercise

1. Evaluate $\int \sec^3 x dx$

$$\begin{aligned}
\int \sec^3 x dx &= \int \sec x d(\tan x) \\
&= \sec x \tan x - \int \tan x d(\sec x) \\
&= \sec x \tan x - \int \tan^2 x \sec x dx \\
&= \sec x \tan x - \int (\sec^2 x - 1) \sec x dx \\
&= \sec x \tan x - \int (\sec^3 x - \sec x) dx \\
&= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx \\
&= \sec x \tan x - \int \sec^3 x dx + \ln |\sec x + \tan x| + C \\
\Rightarrow 2 \int \sec^3 x dx &= \sec x \tan x + \ln |\sec x + \tan x| + C \\
\therefore \int \sec^3 x dx &= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C'
\end{aligned}$$

2. Evaluate $\int \sin^{-1} \frac{x}{a} dx$

$$\begin{aligned}
\int \sin^{-1} \frac{x}{a} dx &= x \sin^{-1} \frac{x}{a} - \int x d(\sin^{-1} \frac{x}{a}) \\
&= x \sin^{-1} \frac{x}{a} - \frac{1}{a} \int \frac{x}{\sqrt{1-x^2/a^2}} dx \\
&= x \sin^{-1} \frac{x}{a} - \frac{1}{a} \int \frac{1}{\sqrt{1-x^2/a^2}} \cdot \frac{d(1-x^2/a^2)}{-2/a^2} \\
&= x \sin^{-1} \frac{x}{a} - \frac{1}{a} \int \left(1 - \frac{x^2}{a^2}\right)^{-1/2} \cdot \frac{d(1-x^2/a^2)}{-2/a^2} \\
&= x \sin^{-1} \frac{x}{a} - \frac{1}{a} \frac{2}{-2/a^2} \left(1 - \frac{x^2}{a^2}\right)^{1/2} + C \\
&= x \sin^{-1} \frac{x}{a} + a \sqrt{1 - \frac{x^2}{a^2}} + C \\
&= x \sin^{-1} \frac{x}{a} + \sqrt{a^2 - x^2} + C
\end{aligned}$$

3. Evaluate $\int x^2 \sin 2x dx$

$$\begin{aligned}
 \int x^2 \sin 2x dx &= -\frac{1}{2} \int x^2 d(\cos 2x) \\
 &= -\frac{1}{2} x^2 \cos 2x + \frac{1}{2} \int \cos 2x d(x^2) \\
 &= -\frac{1}{2} x^2 \cos 2x + \frac{1}{2} \int \cos 2x (2x) dx \\
 &= -\frac{1}{2} x^2 \cos 2x + \int x \frac{d(\sin 2x)}{2} \\
 &= -\frac{1}{2} x^2 \cos 2x + \frac{1}{2} x \sin 2x - \frac{1}{2} \int \sin 2x dx \\
 &= -\frac{1}{2} x^2 \cos 2x + \frac{1}{2} x \sin 2x - \frac{1}{4} \int \sin 2x d(2x) \\
 &= -\frac{1}{2} x^2 \cos 2x + \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x + C \\
 &= \frac{1}{4} (1 - 2x^2) \cos 2x + \frac{1}{2} x \sin 2x + C
 \end{aligned}$$

4. Evaluate $\int e^{2x} \cos^2 3x dx$

$$\begin{aligned}
 \int e^{2x} \cos^2 3x dx &= \frac{1}{2} \int \cos^2 3x d(e^{2x}) \\
 &= \frac{1}{2} e^{2x} \cos^2 3x - \frac{1}{2} \int e^{2x} d(\cos^2 3x) \\
 &= \frac{1}{2} e^{2x} \cos^2 3x - \int e^{2x} \cos 3x (-\sin 3x) (3) dx \\
 &= \frac{1}{2} e^{2x} \cos^2 3x + 3 \int e^{2x} \cos 3x \sin 3x dx \\
 &= \frac{1}{2} e^{2x} \cos^2 3x + \frac{3}{2} \int e^{2x} \cos 6x dx \\
 &= \frac{1}{2} e^{2x} \cos^2 3x + \frac{3}{4} \int \cos 6x d(e^{2x}) \\
 &= \frac{1}{2} e^{2x} \cos^2 3x + \frac{3}{4} e^{2x} \cos 6x - \frac{3}{4} \int e^{2x} d(\cos 6x) \\
 &= \frac{1}{2} e^{2x} \cos^2 3x + \frac{3}{4} e^{2x} \cos 6x + \frac{9}{2} \int e^{2x} \sin 6x dx \\
 &= \frac{1}{2} e^{2x} \cos^2 3x + \frac{3}{4} e^{2x} \cos 6x + \frac{9}{4} \int \sin 6x d(e^{2x}) \\
 &= \frac{1}{2} e^{2x} \cos^2 3x + \frac{3}{4} e^{2x} \cos 6x + \frac{9}{4} e^{2x} \sin 6x - \frac{9}{4} \int e^{2x} d(\sin 6x) \\
 &= \frac{1}{2} e^{2x} \cos^2 3x + \frac{3}{4} e^{2x} \cos 6x + \frac{9}{4} e^{2x} \sin 6x - \frac{27}{2} \int e^{2x} \cos 6x dx \\
 \therefore \int e^{2x} \cos^2 3x dx &= \frac{1}{2} e^{2x} \cos^2 3x + \frac{3}{2} \int e^{2x} \cos 6x dx \\
 &= \frac{1}{2} e^{2x} \cos^2 3x + \frac{3}{4} e^{2x} \cos 6x + \frac{9}{4} e^{2x} \sin 6x - \frac{27}{2} \int e^{2x} \cos 6x dx \\
 \frac{3}{2} \int e^{2x} \cos 6x dx &= \frac{3}{4} e^{2x} \cos 6x + \frac{9}{4} e^{2x} \sin 6x - \frac{27}{2} \int e^{2x} \cos 6x dx \\
 15 \int e^{2x} \cos 6x dx &= \frac{3}{4} e^{2x} \cos 6x + \frac{9}{4} e^{2x} \sin 6x \\
 \int e^{2x} \cos 6x dx &= \frac{1}{20} e^{2x} \cos 6x + \frac{3}{20} e^{2x} \sin 6x \\
 \therefore \int e^{2x} \cos^2 3x dx &= \frac{1}{2} e^{2x} \cos^2 3x + \frac{3}{2} \left(\frac{1}{20} e^{2x} \cos 6x + \frac{3}{20} e^{2x} \sin 6x \right) \\
 &= \frac{1}{2} e^{2x} \cos^2 3x + \frac{3}{40} e^{2x} \cos 6x + \frac{9}{40} e^{2x} \sin 6x
 \end{aligned}$$

5. Evaluate $\int \sqrt{x^2 - a^2} dx$

$$\begin{aligned}
 \int \sqrt{x^2 - a^2} dx &= \int a \sec t \tan t \sqrt{a^2 \sec^2 t - a^2} dt && \text{(sub } x = a \sec t) \\
 &= a^2 \int \sec t \tan^2 t dt \\
 &= a^2 \int \tan t d(\sec t) \\
 &= a^2 \sec t \tan t - a^2 \int \sec t d(\tan t) \\
 &= a^2 \sec t \tan t - a^2 \int \sec^3 t dt \\
 &= a^2 \sec t \tan t - \frac{a^2}{2} \sec t \tan t - \frac{a^2}{2} \ln |\sec t + \tan t| + C' && \text{(use result of \#1)} \\
 &= \frac{a^2}{2} \sec t \tan t - \frac{a^2}{2} \ln |\sec t + \tan t| + C \\
 &= \frac{1}{2} x \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln \left| \frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1} \right| + C \\
 &= \frac{1}{2} x \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln |x + \sqrt{x^2 - a^2}| + C'
 \end{aligned}$$

6. Evaluate $\int \sqrt{a^2 + x^2} dx$

$$\begin{aligned}
 \int \sqrt{a^2 + x^2} dx &= \int a \sec^2 t \sqrt{a^2 \tan^2 t + a^2} dt && \text{(sub } x = a \tan t) \\
 &= a^2 \int \sec^3 t dt \\
 &= \frac{a^2}{2} \sec t \tan t + \frac{a^2}{2} \ln |\sec t + \tan t| + C && \text{(use result of \#1)} \\
 &= \frac{1}{2} x \sqrt{x^2 + a^2} + \frac{a^2}{2} \ln \left| \sqrt{\frac{x^2}{a^2} + 1} + \frac{x}{a} \right| + C \\
 &= \frac{1}{2} x \sqrt{x^2 + a^2} + \frac{a^2}{2} \ln |\sqrt{x^2 + a^2} + x| + C'
 \end{aligned}$$

2 Partial Fractions, and Integration of Rational Functions

2.1 Partial fractions

- Partial fractions:

$$\frac{1}{3x^2 - 2x - 1} = \frac{1/4}{x - 1} + \frac{-3/4}{3x + 1}$$

- In general, a fraction whose numerator and denominator are both polynomial with real coefficients can be expressed as a series of similar fractions, the fractional terms of the series is either degree 0, 1, or 2

- How to do? As follows:

$$\begin{aligned} \frac{1}{3x^2 - 2x - 1} &= \frac{1}{(x-1)(3x+1)} \\ &\equiv \frac{A}{x-1} + \frac{B}{3x+1} && \text{(for some unknowns } A, B) \\ 1 &\equiv A(3x+1) + B(x-1) && \text{(multiply each side by } 3x^2 - 2x - 1) \\ 1 &= A - B && \text{(when } x = 0) \\ 1 &= (3A + B)x + (A - B) \\ \therefore 3A + B &= 0 && \text{(properties of identity)} \\ 3A &= -B \\ A &= \frac{1}{4} && \text{(as } A - B = 1) \\ B &= -\frac{3}{4} \end{aligned}$$

1. Factorize the denominator of the polynomial fraction into products of degree 1 or 2 polynomials
 2. Each factor of the denominator become the denominator of a separate fraction. If there are factors raised to higher powers, each power is a denominator
 3. Numerators are unknown polynomials of a lower degree, to be solved by various method
 4. Sum of them should be identical to the original polynomial fraction
- One of the best way to solve for (numerators of) partial fractions is *the method of undetermined coefficients*

Exercises

1. Express as partial fractions for $\frac{x^2 + x + 1}{x^3 - 4x^2 + x + 6}$

$$\begin{aligned} \frac{x^2 + x + 1}{x^3 - 4x^2 + x + 6} &= \frac{x^2 + x + 1}{(x+1)(x-2)(x-3)} && \text{(factorize)} \\ &\equiv \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x-3} \\ x^2 + x + 1 &= A(x-2)(x-3) + B(x+1)(x-3) + C(x+1)(x-2) \\ 1 &= A(-3)(-4) && \text{(sub } x = -1) \\ \therefore A &= 1/12 \\ 7 &= B(3)(-1) && \text{(sub } x = 2) \\ \therefore B &= -7/3 \\ 13 &= C(4)(1) && \text{(sub } x = 3) \\ \therefore C &= 13/4 \\ \therefore \frac{x^2 + x + 1}{x^3 - 4x^2 + x + 6} &= \frac{1/12}{x+1} + \frac{-7/3}{x-2} + \frac{13/4}{x-3} \\ &= \frac{1}{12(x+1)} - \frac{7}{3(x-2)} + \frac{13}{4(x-3)} \end{aligned}$$

2. Express as partial fractions for $\frac{x^2}{(x+1)(x-1)^3}$

$$\frac{x^2}{(x+1)(x-1)^3} \equiv \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$$

$$x^2 = A(x-1)^3 + B(x+1)(x-1)^2 + C(x+1)(x-1) + D(x+1)$$

$$1 = 2D \quad (\text{sub } x = 1)$$

$$\therefore D = 1/2$$

$$1 = -8A \quad (\text{sub } x = -1)$$

$$\therefore A = -1/8$$

$$2x = 3A(x-1)^2 + B[(x-1)^2 + 2(x+1)(x-1)] + 2Cx + D \quad (\text{differentiate})$$

$$2 = 2C + \frac{1}{2} \quad (\text{sub } x = 1)$$

$$\therefore C = \frac{3}{4}$$

$$0 = \frac{-3}{8} + B[1-2] + \frac{1}{2} \quad (\text{sub } x = 0)$$

$$\therefore B = \frac{1}{8}$$

$$\therefore \frac{x^2}{(x+1)(x-1)^3} = -\frac{1}{8(x+1)} + \frac{1}{8(x-1)} + \frac{3}{4(x-1)^2} + \frac{1}{2(x-1)^3}$$

3. Express as partial fractions for $\frac{3x^3 - 2x - 20}{(x^2 + 3)(2x^2 - 6x + 5)}$

$$\frac{3x^3 - 2x - 20}{(x^2 + 3)(2x^2 - 6x + 5)} \equiv \frac{Ax + B}{x^2 + 3} + \frac{Cx + D}{2x^2 - 6x + 5} \quad (\text{no further factors})$$

$$3x^3 - 2x - 20 = (Ax + B)(2x^2 - 6x + 5) + (Cx + D)(x^2 + 3)$$

$$-20 = 5B + 3D \quad (\text{sub } x = 0)$$

$$3 = 2A + C \quad (\text{coeff. } x^3)$$

$$0 = 2B - 6A + D$$

$$-2 = 5A - 6B + 3C$$

$$\therefore \begin{bmatrix} 0 & 5 & 0 & 3 \\ 2 & 0 & 1 & 0 \\ -6 & 2 & 0 & 1 \\ 5 & -6 & 3 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} -20 \\ 3 \\ 0 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 5 \\ -10 \end{bmatrix}$$

$$\therefore \frac{3x^3 - 2x - 20}{(x^2 + 3)(2x^2 - 6x + 5)} = \frac{-x + 2}{x^2 + 3} + \frac{5x - 10}{2x^2 - 6x + 5}$$

2.2 Integration using Partial Fractions

- Integration of $\int \frac{1}{Ax+B} dx$ can be solved by substituting $y = Ax + B$

$$\begin{aligned}\int \frac{dx}{Ax+B} &= \frac{1}{A} \int \frac{d(Ax+B)}{Ax+B} \\ &= \frac{1}{A} \ln|Ax+B| + C\end{aligned}$$

- Integration of “constant over quadratic”: Competing square!

– Example:

$$\begin{aligned}\int \frac{1}{x^2+2x-3} dx &= \int \frac{1}{(x+1)^2-2^2} dx \\ &= \int \frac{2 \sec t \tan t dt}{2(\sec^2 t - 1)} && \text{(sub } x+1 = 2 \sec t\text{)} \\ &= \int \frac{\sec t}{\tan t} dt \\ &= \int \csc t dt \\ &= \ln|\csc t - \cot t| + C \\ &= \ln \left| \frac{x+1}{\sqrt{(x+1)^2-2}} - \frac{2}{\sqrt{(x+1)^2-2}} \right| + C \\ &= \ln \left| \frac{x-1}{\sqrt{x^2+2x-3}} \right| + C\end{aligned}$$

- Integration of “linear over quadratic”: Break into two!

– Example:

$$\begin{aligned}\int \frac{4x+5}{x^2+2x-3} dx &= \int \frac{2(2x+2)+1}{x^2+2x-3} dx \\ &= 2 \int \frac{2x+2}{x^2+2x-3} dx + \int \frac{1}{x^2+2x-3} dx \\ &= 2 \int \frac{d(x^2+2x-3)}{x^2+2x-3} + \int \frac{1}{x^2+2x-3} dx \\ &= 2 \ln|x^2+2x-3| + \ln \left| \frac{x-1}{\sqrt{x^2+2x-3}} \right| + C\end{aligned}$$

- Other kinds of fractions: Partial fractions!

• Example:

$$\begin{aligned}
 \int \frac{x^5 + x^3 - 1}{x^2(x^2 + 1)^2} dx &= \int \left(\frac{x^5}{x^2(x^2 + 1)^2} + \frac{x^3}{x^2(x^2 + 1)^2} - \frac{1}{x^2(x^2 + 1)^2} \right) dx \\
 &= \int \frac{x^3}{(x^2 + 1)^2} dx + \int \frac{x}{(x^2 + 1)^2} dx - \int \frac{1}{x^2(x^2 + 1)^2} dx \\
 \int \frac{x^3}{(x^2 + 1)^2} dx &= \int \frac{x}{x^2 + 1} dx - \int \frac{x}{(x^2 + 1)^2} dx \\
 &= \frac{1}{2} \ln|x^2 + 1| + \frac{1}{2} \cdot \frac{1}{x^2 + 1} + C_1 \\
 \int \frac{x}{(x^2 + 1)^2} dx &= -\frac{1}{2} \cdot \frac{1}{x^2 + 1} + C_2 \quad \text{(see above!)} \\
 \int \frac{dx}{x^2(x^2 + 1)^2} &= \int \frac{1}{x^2} dx + \int \frac{-1}{x^2 + 1} dx + \int \frac{-1}{(x^2 + 1)^2} dx \\
 &= -\frac{1}{x} - \int \frac{1}{x^2 + 1} dx - \int \frac{(x^2 + 1) - x^2}{(x^2 + 1)^2} dx \\
 &= -\frac{1}{x} - \int \frac{1}{x^2 + 1} dx - \int \frac{1}{x^2 + 1} dx + \int \frac{x^2}{(x^2 + 1)^2} dx \\
 &= -\frac{1}{x} - 2 \int \frac{1}{x^2 + 1} dx + \frac{-1}{2} \int x d\left(\frac{1}{x^2 + 1}\right) \\
 &= -\frac{1}{x} - 2 \tan^{-1} x - \frac{1}{2} \frac{x}{x^2 + 1} + \frac{1}{2} \int \frac{dx}{x^2 + 1} \\
 &= -\frac{1}{x} - 2 \tan^{-1} x - \frac{1}{2} \frac{x}{x^2 + 1} + \frac{1}{2} \tan^{-1} x + C_3 \\
 &= -\frac{1}{x} - \frac{3}{2} \tan^{-1} x - \frac{1}{2} \frac{x}{x^2 + 1} + C_3 \\
 \therefore \int \frac{x^5 + x^3 - 1}{x^2(x^2 + 1)^2} dx &= \frac{1}{2} \ln|x^2 + 1| + \frac{1}{2} \cdot \frac{1}{x^2 + 1} + C_1 - \frac{1}{2} \cdot \frac{1}{x^2 + 1} + C_2 + \frac{1}{x} + \frac{3}{2} \tan^{-1} x + \frac{1}{2} \frac{x}{x^2 + 1} - C_3 \\
 &= \frac{1}{2} \ln(x^2 + 1) + \frac{3}{2} \tan^{-1} x + \frac{3x^2 + 2}{2x(x^2 + 1)} + C
 \end{aligned}$$

Exercises

1. Evaluate $\int \frac{x^2 + x + 1}{x^3 - 4x^2 + x + 6} dx$

$$\begin{aligned}
 \int \frac{x^2 + x + 1}{x^3 - 4x^2 + x + 6} dx &= \int \left(\frac{1}{12(x+1)} - \frac{7}{3(x-2)} + \frac{13}{4(x-3)} \right) dx \\
 &= \frac{1}{12} \int \frac{dx}{x+1} - \frac{7}{3} \int \frac{dx}{x-2} + \frac{13}{4} \int \frac{dx}{x-3} \\
 &= \frac{1}{12} \ln|x+1| - \frac{7}{3} \ln|x-2| + \frac{13}{4} \ln|x-3| + C \\
 &= \frac{1}{12} \ln \left| \frac{(x+1)(x-3)^{39}}{(x-2)^{28}} \right| + C
 \end{aligned}$$

2. Evaluate $\int \frac{x^2}{(x+1)(x-1)^3} dx$

$$\begin{aligned} \int \frac{x^2}{(x+1)(x-1)^3} dx &= \int \left(-\frac{1}{8(x+1)} + \frac{1}{8(x-1)} + \frac{3}{4(x-1)^2} + \frac{1}{2(x-1)^3} \right) dx \\ &= -\frac{1}{8} \int \frac{dx}{x+1} + \frac{1}{8} \int \frac{dx}{x-1} + \frac{3}{4} \int \frac{dx}{(x-1)^2} + \frac{1}{2} \int \frac{dx}{(x-1)^3} \\ &= -\frac{1}{8} \ln|x+1| + \frac{1}{8} \ln|x-1| - \frac{3}{4} \cdot \frac{1}{x-1} - \frac{1}{4} \cdot \frac{1}{(x-1)^2} + C \\ &= \frac{1}{8} \ln \left| \frac{x-1}{x+1} \right| + \frac{2-3x}{4(x-1)^2} + C \end{aligned}$$

3. Evaluate $\int \frac{3x^3 - 2x - 20}{(x^2 + 3)(2x^2 - 6x + 5)} dx$

$$\begin{aligned} \int \frac{3x^3 - 2x - 20}{(x^2 + 3)(2x^2 - 6x + 5)} dx &= \int \left(\frac{-x+2}{x^2+3} + \frac{5x-10}{2x^2-6x+5} \right) dx \\ &= -\int \frac{x}{x^2+3} dx + \int \frac{2}{x^2+3} dx + \int \frac{5x-10}{2x^2-6x+5} dx \\ &= -\frac{1}{2} \int \frac{2x}{x^2+3} dx + 2 \int \frac{1}{x^2+(\sqrt{3})^2} dx + \frac{5}{4} \int \frac{4x-6}{2x^2-6x+5} dx - \frac{5}{4} \int \frac{2}{2x^2-6x+5} dx \\ &= -\frac{1}{2} \ln|x^2+3| + 2 \left(\frac{1}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} \right) + \frac{5}{4} \ln|2x^2-6x+5| - \frac{5}{4} \int \frac{dx}{(x-\frac{3}{2})^2 + (\frac{1}{2})^2} \\ &= -\frac{1}{2} \ln|x^2+3| + 2 \left(\frac{1}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} \right) + \frac{5}{4} \ln|2x^2-6x+5| - \frac{5}{4} \left(\frac{1}{1/2} \tan^{-1} \frac{x-3/2}{1/2} \right) + C \\ &= -\frac{1}{2} \ln|x^2+3| + 2 \left(\frac{1}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} \right) + \frac{5}{4} \ln|2x^2-6x+5| - \frac{5}{4} (2 \tan^{-1}(2x-3)) + C \\ &= \frac{1}{4} \ln \left| \frac{(2x^2-6x+5)^5}{(x^2+3)^2} \right| + \frac{2}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} - \frac{5}{2} \tan^{-1}(2x-3) + C \end{aligned}$$

3 Bring-home Practices

1. $\int \frac{(3x-1)}{x^2+9} dx$
2. $\int \frac{x}{\sqrt{27+6x-x^2}} dx$
3. $\int \frac{x}{(x+1)(x+3)(x+5)} dx$
4. $\int \frac{x^5-x^3+1}{x^4-x^3} dx$
5. $\int \sec^5 x dx$
6. $\int \tan^5 x dx$
7. $\int \frac{dx}{x\sqrt{x^2+3}}$
8. $\int \frac{xdx}{\sqrt{3+2x-x^2}}$
9. $\int e^{ax} \sin bxdx$