Remedial Lesson 5: Definite Integrals

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1 Definition of Definite Integral

- Definite integral as limit of sum
- Riemann Integral:

$$\int_a^b f(x)dx = \lim_{||x||\to 0}\sum_{k=0}^{n} f(x_k)(x_{k+1}-x_k)$$
$$\text{where} \quad a = x_0 < x_1 < \cdots < x_n = b$$
$$||x|| = \max_k (x_{k+1}-x_k)$$

- Riemann-Stieltjes Integral:

$$\int_a^b f(x)dG(x) = \lim_{||x||\to 0}\sum_{k=0}^{n} f(x_k)[G(x_{k+1})-G(x_k)]$$
$$\text{where} \quad a = x_0 < x_1 < \cdots < x_n = b$$
$$||x|| = \max_k (x_{k+1}-x_k)$$

$$\int_a^b f(x)dG(x) = \int_a^b f(x)g(x)dx$$

- Newton-Leibniz Formula:

$$\int_a^b f(x)dx = \left[\int f(x)dx\right]_a^b = F(b)-F(a)$$

  - Constant of integration is ignored (as it will be cancelled eventually)

Examples

1. Find the area under the curve $y = x^2$ from $x = 0$ to $x = 1$

$$\int_0^1 x^2 dx$$
$$= \left[\frac{1}{3}x^3\right]_0^1$$
$$= \frac{1}{3}$$

2. Find the are under the curve $y = 2x + 1$ from $x = 0$ to $x = 2$

$$\begin{aligned}
&\int_0^2 (2x+1)dx && \text{(trapezeum) Left base} = 1\\
&= \left[\int (2x+1)dx\right]_0^2 && \text{Right base} = 5\\
&= \left[x^2 + x\right]_0^2 && \text{Height} = 2\\
&= 2^2 + 2 && \text{Area} = \frac{1}{2}(1+5)(2)\\
&= 6 && = 6
\end{aligned}$$

3. Find the area of half unit-circle $y = \sqrt{1 - x^2}$

$$\begin{aligned}
&\min\ x = -1; \quad \max\ x = +1\\
\therefore\ \text{area} &= \int_{-1}^{1} \sqrt{1 - x^2}dx\\
&= \int_{x=-1}^{x=1} \sqrt{1 - \sin^2 t}d(\sin t)\\
&= \int_{-\pi/2}^{\pi/2} \cos^2 t dt\\
&= \int_{-\pi/2}^{\pi/2} \frac{1 + \cos 2t}{2} dt\\
&= \left[\frac{1}{2}t + \frac{1}{4}\sin 2t\right]_{-\pi/2}^{\pi/2}\\
&= \frac{1}{2} \cdot \frac{\pi}{2} - \frac{1}{2}\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}
\end{aligned}$$

Exercises

1. Find the area of the semicircle: $y = \sqrt{r^2 - x^2}$

2. Find the area between the x-axis and the upper half of the ellipse $a^2x^2 + b^2y^2 = r^2$

2 Properties of Definite Integral

- Definite integral is the limit of sum / area under the curve
- Area enclosed by a counter-clockwise path is positive, otherwise it is negative
  - Example: $\int_0^{\pi} \sin x dx > 0$ but $\int_{\pi}^{2\pi} \sin x dx < 0$
  - In terms of the area under the curve $y = f(x)$, the area is positive if $f(x) > 0$, and negative if $f(x) < 0$
  - Integration from right to left reversed the sign

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

- Area equals to the sum of sub-area

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

  - The above formula is true for both $a < b < c$ and $a < c < b$

- Rule of substitution: If $x = g(t)$ is used for substitution,

$$\int_a^b f(x)dx = \int_{\alpha}^{\beta} f(g(t))g'(t)dt$$

$$\text{where } a = g(\alpha)$$
$$b = g(\beta)$$

- Dummy variable: The variable used in definite integral is unimportant,

$$\int_a^b f(x)dx = \int_a^b f(t)dt$$

Exercises

1. Prove $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

2. Prove $\int_0^\pi x f(\sin x)dx = \frac{\pi}{2}\int_0^\pi f(\sin x)dx$ (Hint: Prove their difference is zero)

3. Evaluate $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx$

4. Simplify and find the derivative of $g(x) = \dfrac{\sqrt{1+\cos x}}{\sqrt{1+\cos x}+\sqrt{1-\cos x}}$ and hence find $\int_0^{\pi} g(x)dx$

3 Helpful Knowledge

- Odd function means for all x, we have $f(-x) = -f(x)$
  - Example: $\sin x$
  - For integration of the odd function, $\int_{-a}^{a} f(x)dx = 0$
- Even function means for all x, we have $f(-x) = f(x)$
  - Example: $\cos x$
  - For integration of the even function, $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

- Periodic function with period T means for all x, we have $f(x) = f(x+T)$
  - Example: $\sin x$ has period of 2π
  - For integration of the periodic function with period T, we have
    1. $\int_a^b f(x)dx = \int_{a+T}^{b+T} f(x)dx$
    2. $\int_0^T f(x)dx = \int_a^{a+T} f(x)dx$
    3. $\int_0^{nT} f(x)dx = n\int_0^T f(x)dx$

Exercises

1. Evaluate $\int_{-1}^{1}(e^x + e^{-x})\sin x dx$

2. Evaluate $\int_{-1}^{1}(e^x - e^{-x})\cos x dx$

3. Evaluate $\int_{6\pi/7}^{20\pi/7}\sin x dx$

4 Integrated Exercises

1. $\int_0^1 e^{\sqrt{x}}dx$

2. $\int_1^2 \frac{e^{2x}}{e^x - 1} dx$

3. $\int_0^{\pi/3} x \sin 3x dx$

4. $\int_{-1}^4 f(x) dx$ where $f(x) = \begin{cases} -2x & \text{if } x \leq 0 \\ \frac{1}{2}x & \text{if } 0 < x \leq 2 \\ 2x - 3 & \text{if } x > 2 \end{cases}$

5. $\int_0^\infty xe^{-x}dx$

6. $\int_0^1 \frac{1}{\sqrt{x}}dx$

7. $\int_0^a \frac{x}{\sqrt{a^2-x^2}}dx$

8. Given $I_n = \int_0^{\pi/4} \sec^n x dx$, Express I_3 in terms of I_1