Remedial Lesson 6: Application of Calculus I

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1 Find area using integration

- Given the curve $y = f(x)$ in Cartesian coordinates, the area under the curve from $x = a$ to $x = b$ is given by

$$A = \int_a^b f(x)dx$$

- Given the curve $r = f(\theta)$ in polar coordinates, the area bounded by the curve and the radial vectors $\theta = a$ and $\theta = b$ is given by

$$A = \frac{1}{2}\int_a^b r^2 d\theta$$

- Actually, polar coordinate and Cartesian coordinate can be interchanged:

$$\begin{aligned} r^2 &= x^2 + y^2 & x &= r\cos\theta \\ \theta &= \tan^{-1}\frac{y}{x} & y &= r\sin\theta \end{aligned}$$

- Example: Find the area of circle with radius r in Cartesian coordinate

$$\begin{aligned}
\text{Equation: } x^2 + y^2 &= r^2 \\
\therefore \quad y^2 &= r^2 - x^2 \\
\therefore \quad A &= 2\int_{-r}^{r} \sqrt{r^2 - x^2}dx \\
&= 2\int_{-r}^{r} \sqrt{r^2 - r^2\sin^2 t}d(r\sin t) && (\text{sub } x = r\sin t) \\
&= 2\int_{-\pi/2}^{\pi/2} r^2\sqrt{1-\sin^2 t}\cos t dt \\
&= 2r^2\int_{-\pi/2}^{\pi/2} \cos^2 t dt \\
&= 2r^2\int_{-\pi/2}^{\pi/2} \frac{1+\cos 2t}{2}dt \\
&= r^2\left[t + \frac{1}{2}\sin 2t\right]_{-\pi/2}^{\pi/2} \\
&= \pi r^2
\end{aligned}$$

- Example: Find the area of circle with radius r in Polar coordinate

$$\begin{aligned} A &= \frac{1}{2}\int_0^{2\pi} r^2 d\theta \\ &= \frac{1}{2}r^2\int_0^{2\pi}(1)d\theta \\ &= \frac{1}{2}r^2\left[\theta\right]_0^{2\pi} \\ &= \pi r^2 \end{aligned}$$

- Example: Find $\int_0^\infty e^{-x^2}dx$

$$\begin{aligned} &\int_0^\infty e^{-x^2}dx \\ =&\sqrt{\int_0^\infty e^{-x^2}dx\cdot\int_0^\infty e^{-y^2}dy} \\ =&\sqrt{\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy} \\ =&\sqrt{\int_0^\infty\int_0^{\pi/2} e^{-r^2}rd\theta dr} \\ =&\sqrt{\int_0^\infty\int_0^{\pi/2} d\theta e^{-r^2}rdr} \\ =&\sqrt{\frac{\pi}{2}\int_0^\infty re^{-r^2}dr} \\ =&\sqrt{\frac{\pi}{4}\int_0^\infty e^{-r^2}dr^2} \\ =&\sqrt{\frac{\pi}{4}\left[-e^{-r^2}\right]_0^\infty} \\ =&\sqrt{\frac{\pi}{4}[0-(-1)]} \\ =&\frac{\sqrt{\pi}}{2} \end{aligned}$$

$$\begin{cases} x^2+y^2 &= r^2 \\ dxdy &= \frac{1}{2}d(r^2)d\theta \end{cases}$$

(integration of first quadrant)

2 Find limit using L'Hôpital's Rule

- Limit means the value of a function as the variable approaches a value
  - Example: As x tends to 1, $f(x) = x+1$ tends to 2, i.e. $\lim_{x\to 1} f(x) = 2$
  - Example:

$$\lim_{x\to 2}\frac{x^2-4x+4}{x-2} = \lim_{x\to 2}\frac{(x-2)^2}{x-2} = \lim_{x\to 2}(x-2) = 0$$

- We usually interested at the limit towards ∞, $-\infty$, and 0

$$\begin{aligned} \lim_{x\to\infty}\frac{x^2+2}{x} &= \infty \\ \lim_{x\to-\infty}\frac{x+1}{x+2} &= 1 \\ \lim_{x\to 0}\frac{x+1}{x^2} &= \infty \end{aligned}$$

- Sometimes, we cannot find the limit so easily, so we have the l'Hôpital's rule:

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

this is applicable when the direct substitution have the undeterminate forms: $\frac{\infty}{\infty}$, $\frac{0}{0}$, $0\cdot\infty$, $\infty-\infty$, 0^0, ∞^0, 1^∞

- Example:

$$\begin{aligned}\lim_{x\to 1}\frac{x-1}{\sqrt{x^2-1}}&=\lim_{x\to 1}\frac{1}{2x\Big/2\sqrt{x^2-1}}\\&=\lim_{x\to 1}\frac{\sqrt{x^2-1}}{x}\\&=\frac{0}{1}\\&=0\end{aligned}$$

- Example:

$$\begin{aligned}\lim_{x\to 0}\frac{\tan x}{\tan 3x}&=\lim_{x\to 0}\frac{\sec^2 x}{3\sec^2 3x}\\&=\frac{1}{3}\end{aligned}$$

- Example:

$$\begin{aligned}\lim_{x\to 0}\tan x\ln x&=\lim_{x\to 0}\frac{\ln x}{\cot x}\\&=\lim_{x\to 0}\frac{1/x}{-\csc^2 x}\\&=\lim_{x\to 0}\frac{-\sin^2 x}{x}\\&=\lim_{x\to 0}\frac{-2\sin x\cos x}{1}\\&=0\end{aligned}$$

- Example:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin x}{x}&=\lim_{x\to 0}\frac{\cos x}{1}\\&=\frac{1}{1}\\&=1\end{aligned}$$

3 Verify Series Convergence by Integration

- Given the monotonically decreasing function $f(x)$, the infinite series, $\sum_{x=k}^{\infty} f(x)$ is bounded (i.e. not infinitely large), if and only if $\int^{\infty} f(x)dx$ also bounded (evaluate only the upper limit)

- Example:

$$\begin{aligned} \frac{1}{x} &> \frac{1}{x+1} && \therefore \text{ decreasing} \\ \int^{\infty} \frac{1}{x}dx &= [\ln x]^{\infty} \\ &= \ln\infty \\ &= \infty \\ \therefore \quad \sum_{x=1}^{\infty} \frac{1}{x} &= \infty && \text{i.e. diverging series} \end{aligned}$$

- Example:

$$\begin{aligned} \frac{1}{x^2} &> \frac{1}{(x+1)^2} && \therefore \text{ decreasing} \\ \int^{\infty} \frac{1}{x^2}dx &= \left[\frac{-1}{x}\right]^{\infty} \\ &= 0 \\ &< \infty \\ \therefore \quad \sum_{x=1}^{\infty} \frac{1}{x^2} &< \infty && \text{i.e. converging} \end{aligned}$$

- Example: Check for the convergence of $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$

$$\begin{aligned} \int^{\infty} \frac{x^2}{x^3+1}dx &= \int^{\infty} \frac{d(x^3)}{3(x^3+1)} \\ &= \left[\frac{1}{3}\ln(x^3+1)\right]^{\infty} \\ &= \infty \\ \therefore \quad \sum_{n=1}^{\infty} \frac{n^2}{n^3+1} &= \infty \quad \text{(diverging)} \end{aligned}$$

- Example: Check for the convergence of $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+9}}$

$$\begin{aligned} \int^{\infty} \frac{1}{\sqrt{x^2+9}}dx &= \int^{\pi/2} \frac{3\sec^2 t\,dt}{\sqrt{9\tan^2 t+9}} && \text{(sub } x = 3\tan t\text{)} \\ &= \int^{\pi/2} \sec t\,dt \\ &= [\ln|\sec t + \tan t|]^{\pi/2} \\ &= \ln|\infty| \\ &= \infty \\ \therefore \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+9}} & \text{ is diverging} \end{aligned}$$

4 Approximation using Differentials

- Differential:

$$dy = f(x)dx$$

- Hence we can approximate the derivation of y by

$$\begin{aligned} f(x+\Delta x) &= y + \Delta y \\ &\approx f(x) + f'(x)\Delta x \end{aligned}$$

- This is the basis for "small perturbation analysis" and why we need to study linear systems in detail
- Example: Find a **good** approximate of $\sqrt{4.1}$ without using calculator

$$\begin{aligned} \frac{d}{dx}\sqrt{x} &= \frac{1}{2\sqrt{x}} \\ \therefore \quad \sqrt{4+0.1} &\approx \sqrt{4} + \frac{1}{2\sqrt{4}}(0.1) \\ &= 2 + \frac{1}{4}(0.1) \\ &= 2.025 \\ \text{Actually, } \sqrt{4.1} &= 2.02484567... \end{aligned}$$

- Example: Find a good approximate of $\sqrt[3]{7}$ without using calculator

$$\begin{aligned} \frac{d}{dx}\sqrt[3]{x} &= \frac{1}{3\sqrt[3]{x^2}} \\ \therefore \quad \sqrt[3]{7} &= \sqrt[3]{8-1} \\ &\approx \sqrt[3]{8} + \frac{1}{3\sqrt[3]{8^2}}(-1) \\ &= 2 + \frac{1}{3(2^2)}(-1) \\ &= 2 - \frac{1}{12} \\ &= 2 - 0.08333... \\ &= 1.91666... \\ \text{Actually, } \sqrt[3]{7} &= 1.9129311... \end{aligned}$$

- Example: Find a good approximate of π^2

$$\begin{aligned} \frac{d}{dx}x^2 &= 2x \\ \pi^2 &= (3.1415926...)^2 \\ &\approx 3^2 + 2(3)(0.1415926...) \\ &= 9 + 6(0.1415926...) \\ &= 9.849555... \\ \text{Actually, } \pi^2 &= 9.8696044... \end{aligned}$$

5 Using integration to solve differential equations

- Differential equations is the equation involving derivatives of functions
- Example:

$$f(x)+\frac{d}{dx}f(x)=x^2+2x$$

- Solving differential equation means finding out the function, for example, the solution for the above equation is $f(x)=x^2$.
- The easiest form of differential equation is the separable equation, namely, we can write the equation in the form:

$$g(y)\frac{dy}{dx}=f(x)$$

which can be solved by:

$$\begin{aligned} g(y)\frac{dy}{dx} &= f(x) \\ g(y)dy &= f(x)dx \\ \therefore \quad \int g(y)dy &= \int f(x)dx \end{aligned}$$

- Example: Solve y for $\frac{dy}{dx}=\frac{y^2}{1+x^2}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{y^2}{1+x^2} \\ \therefore \quad \frac{1}{y^2}dy &= \frac{1}{1+x^2}dx \\ \int \frac{1}{y^2}dy &= \int \frac{1}{1+x^2}dx \\ -\frac{1}{y} &= \tan^{-1}x+C \\ y &= \frac{-1}{\tan^{-1}x+C} \end{aligned}$$

- Example: Solve y for $\frac{dy}{dx}=\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$, given $y=1$ when $x=0$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} \\ \int \frac{1}{\sqrt{1-y^2}}dy &= \int \frac{1}{\sqrt{1-x^2}}dx \\ \sin^{-1}y &= \sin^{-1}x+C \\ \because \quad \sin^{-1}1 &= \sin^{-1}0+C \\ \frac{\pi}{2} &= C \\ \therefore \quad \sin^{-1}y &= \sin^{-1}x+\frac{\pi}{2} \\ y &= \sin(\sin^{-1}x)\cos\frac{\pi}{2}+\cos(\sin^{-1}x)\sin\frac{\pi}{2} \\ &= \cos\sin^{-1}x \\ &= \sqrt{1-x^2} \end{aligned}$$

- Example: A stationary particle of mass m fall under gravity. When it has velocity v, it experiences a resistance force $f(v) = -2v$. Express displacement s in terms of time t.

$$
\begin{aligned}
v &= \frac{dx}{dt} \\
v &= u + \int a(t)dt \\
&= \int a(t)dt \\
\text{Acceleration satisfies: } F &= ma \\
mg - 2v &= ma \\
a &= \frac{mg-2v}{m} \\
\therefore \quad v &= \int \left(\frac{mg-2v}{m} \right) dt \\
mv &= mgt - 2\int vdt \\
&= mgt - 2s \\
m\frac{ds}{dt} &= mgt - 2s \\
ms &= e^{-2t} \int e^{2t} mgt dt \\
&= \frac{1}{2} e^{-2t} \left[e^{2t} mgt - mg \int e^{2t} dt \right] \\
&= \frac{1}{2} e^{-2t} \left[e^{2t} mgt - \frac{1}{2} e^{2t} mg + C \right] \\
s &= \frac{1}{2} gt - \frac{1}{4} g + C' e^{-2t} \\
0 &= \frac{1}{2} gt - \frac{1}{4} g + C' \\
C' &= \frac{1}{4} g \\
s &= \frac{1}{4} g(2t + e^{-2t} - 1)
\end{aligned}
$$

6 Maclaurin Series and Taylor Series

- Maclaurin Series is to express *any* function $f(x)$ as the infinite power series:

$$
\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} \frac{x^k}{k!} f^{(k)}(0) \\
&= f(0) + xf'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + \cdots
\end{aligned}
$$

- Taylor Series is a generalization of Maclaurin Series:

$$
\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!} f^{(k)}(a) \\
&= f(a) + (x-a)f'(0) + \frac{(x-a)^2}{2} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots + \frac{(x-a)^n}{n!} f^{(n)}(a) + \cdots
\end{aligned}
$$

  so we usually call Maclaurin series as Taylor series.

- With Taylor series, everything can be expressed as polynomial

- Example: Express e^x as Taylor series

$$\begin{aligned} e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} f^{(k)}(0) \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!} e^0 \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \end{aligned}$$

- Example: Express $\sin x$ as Taylor series

$$\begin{aligned} \sin x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} f^{(k)}(0) \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \sin(0 + \frac{k\pi}{2}) \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \end{aligned}$$

- Example: Express $\cos x$ as Taylor series

$$\begin{aligned} \cos x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} f^{(k)}(0) \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \cos(0 + \frac{k\pi}{2}) \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \end{aligned}$$

- Example: Express $\frac{1}{x+1}$ as Taylor series

$$\begin{aligned} \frac{1}{x+1} &= \sum_{k=0}^{\infty} \frac{x^k}{k!} f^{(k)}(0) \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \cdot \frac{d^k}{dx^k} (x+1)^{-1} \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \left[(-1)^k k! (x+1)^{-1-k} \Big|_{x=0} \right] \\ &= \sum_{k=0}^{\infty} (-1)^k x^k \\ &= 1 - x + x^2 - x^3 + x^4 - \cdots \end{aligned}$$