

# Remedial Lesson 6: Application of Calculus I

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## 1 Find area using integration

- Given the curve  $y = f(x)$  in Cartesian coordinates, the area under the curve from  $x = a$  to  $x = b$  is given by

$$A = \int_a^b f(x) dx$$

- Given the curve  $r = f(\theta)$  in polar coordinates, the area bounded by the curve and the radial vectors  $\theta = a$  and  $\theta = b$  is given by

$$A = \frac{1}{2} \int_a^b r^2 d\theta$$

- Actually, polar coordinate and Cartesian coordinate can be interchanged:

$$r^2 = x^2 + y^2$$

$$x = r \cos \theta$$

$$\theta = \tan^{-1} \frac{y}{x}$$

$$y = r \sin \theta$$

- Example: Find the area of circle with radius  $r$  in Cartesian coordinate

$$\text{Equation: } x^2 + y^2 = r^2$$

$$\therefore y^2 = r^2 - x^2$$

$$\therefore A = 2 \int_{-r}^r \sqrt{r^2 - x^2} dx$$

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(sub  $x = r \sin t$ )

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$$= \pi r^2$$

- Example: Find the area of circle with radius  $r$  in Polar coordinate

$$\begin{aligned} A &= \frac{1}{2} \int_0^{2\pi} r^2 d\theta \\ &= \\ &= \\ &= \pi r^2 \end{aligned}$$

- Example: Find  $\int_0^\infty e^{-x^2} dx$

$$\begin{aligned} &\int_0^\infty e^{-x^2} dx \\ &= \sqrt{\int_0^\infty e^{-x^2} dx \cdot \int_0^\infty e^{-y^2} dy} \\ &= \sqrt{\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy} && \begin{cases} x^2 + y^2 = r^2 \\ dx dy = \frac{1}{2} d(r^2) d\theta \end{cases} \\ &= \sqrt{\int_0^\infty \int_0^{\pi/2} e^{-r^2} r d\theta dr} && \text{(integration of first quadrant)} \\ &= \sqrt{\int_0^\infty \int_0^{\pi/2} d\theta e^{-r^2} r dr} \\ &= \sqrt{\frac{\pi}{2} \int_0^\infty r e^{-r^2} dr} \\ &= \sqrt{\frac{\pi}{4} \int_0^\infty e^{-r^2} dr^2} \\ &= \sqrt{\frac{\pi}{4} [-e^{-r^2}]_0^\infty} \\ &= \sqrt{\frac{\pi}{4} [0 - (-1)]} \\ &= \frac{\sqrt{\pi}}{2} \end{aligned}$$

## 2 Find limit using L'Hôpital's Rule

- Limit means the value of a function as the variable approaches a value

– Example: As  $x$  tends to 1,  $f(x) = x + 1$  tends to 2, i.e.  $\lim_{x \rightarrow 1} f(x) = 2$

– Example:

$$\lim_{x \rightarrow 2} \frac{x^2 - 4x + 4}{x - 2} = \lim_{x \rightarrow 2} \frac{(x - 2)^2}{x - 2} = \lim_{x \rightarrow 2} (x - 2) = 0$$

- We usually interested at the limit towards  $\infty$ ,  $-\infty$ , and 0

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x^2 + 2}{x} &= \infty \\ \lim_{x \rightarrow -\infty} \frac{x + 1}{x + 2} &= 1 \\ \lim_{x \rightarrow 0} \frac{x + 1}{x^2} &= \infty \end{aligned}$$

- Sometimes, we cannot find the limit so easily, so we have the l'Hôpital's rule:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

this is applicable when the direct substitution have the undeterminate forms:  $\frac{\infty}{\infty}$ ,  $\frac{0}{0}$ ,  $0 \cdot \infty$ ,  $\infty - \infty$ ,  $0^0$ ,  $\infty^0$ ,  $1^\infty$

- Example:

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x-1}{\sqrt{x^2-1}} &= \\ &= \\ &= \frac{0}{1} \\ &= 0 \end{aligned}$$

- Example:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\tan x}{\tan 3x} &= \\ &= \end{aligned}$$

- Example:

$$\begin{aligned} \lim_{x \rightarrow 0} \tan x \ln x &= \\ &= \\ &= \\ &= \end{aligned}$$

- Example:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin x}{x} &= \\ &= \\ &= 1 \end{aligned}$$

### 3 Verify Series Convergence by Integration

- Given the monotonically decreasing function  $f(x)$ , the infinite series,  $\sum_{x=k}^{\infty} f(x)$  is bounded (i.e. not infinitely large), if and only if  $\int^{\infty} f(x)dx$  also bounded (evaluate only the upper limit)

- Example:

$$\begin{aligned} \frac{1}{x} &> \frac{1}{x+1} && \therefore \text{decreasing} \\ \int^{\infty} \frac{1}{x} dx &= [\ln x]^{\infty} \\ &= \ln \infty \\ &= \infty \\ \therefore \sum_{x=1}^{\infty} \frac{1}{x} &= \infty && \text{i.e. diverging series} \end{aligned}$$

- Example:

$$\begin{aligned} \frac{1}{x^2} &> \frac{1}{(x+1)^2} && \therefore \text{decreasing} \\ \int^{\infty} \frac{1}{x^2} dx &= \left[ \frac{-1}{x} \right]^{\infty} \\ &= 0 \\ &< \infty \\ \therefore \sum_{x=1}^{\infty} \frac{1}{x^2} &< \infty && \text{i.e. converging} \end{aligned}$$

- Example: Check for the convergence of  $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$

$$\begin{aligned} \int^{\infty} \frac{x^2}{x^3+1} dx &= \int^{\infty} \frac{d(x^3)}{3(x^3+1)} \\ &= \\ &= \infty \\ \therefore \sum_{n=1}^{\infty} \frac{n^2}{n^3+1} &= \infty \quad (\text{diverging}) \end{aligned}$$

- Example: Check for the convergence of  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+9}}$

## 4 Approximation using Differentials

- Differential:

$$dy = f'(x)dx$$

- Hence we can approximate the derivation of  $y$  by

$$\begin{aligned}f(x + \Delta x) &= y + \Delta y \\ &\approx f(x) + f'(x)\Delta x\end{aligned}$$

- This is the basis for “small perturbation analysis” and why we need to study linear systems in detail
- Example: Find a **good** approximate of  $\sqrt{4.1}$  without using calculator

$$\begin{aligned}\frac{d}{dx}\sqrt{x} &= \frac{1}{2\sqrt{x}} \\ \therefore \sqrt{4+0.1} &\approx \sqrt{4} + \frac{1}{2\sqrt{4}}(0.1) \\ &= 2 + \frac{1}{4}(0.1) \\ &= 2.025\end{aligned}$$

Actually,  $\sqrt{4.1} = 2.02484567\dots$

- Example: Find a good approximate of  $\sqrt[3]{7}$  without using calculator

- Example: Find a good approximate of  $\pi^2$

## 5 Using integration to solve differential equations

- Differential equations is the equation involving derivatives of functions

- Example:

$$f(x) + \frac{d}{dx}f(x) = x^2 + 2x$$

- Solving differential equation means finding out the function, for example, the solution for the above equation is  $f(x) = x^2$ .
- The easiest form of differential equation is the separable equation, namely, we can write the equation in the form:

$$g(y)\frac{dy}{dx} = f(x)$$

which can be solved by:

$$\begin{aligned} g(y)\frac{dy}{dx} &= f(x) \\ g(y)dy &= f(x)dx \\ \therefore \int g(y)dy &= \int f(x)dx \end{aligned}$$

- Example: Solve y for  $\frac{dy}{dx} = \frac{y^2}{1+x^2}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{y^2}{1+x^2} \\ \therefore \frac{1}{y^2}dy &= \frac{1}{1+x^2}dx \\ \int \frac{1}{y^2}dy &= \int \frac{1}{1+x^2}dx \\ -\frac{1}{y} &= \tan^{-1}x + C \\ y &= \frac{-1}{\tan^{-1}x + C} \end{aligned}$$

- Example: Solve y for  $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$ , given  $y = 1$  when  $x = 0$

- Example: A stationary particle of mass  $m$  fall under gravity. When it has velocity  $v$ , it experiences a resistance force  $f(v) = -2v$ . Express displacement  $s$  in terms of time  $t$ .

## 6 Maclaurin Series and Taylor Series

- Maclaurin Series is to express *any* function  $f(x)$  as the infinite power series:

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{x^k}{k!} f^{(k)}(0) \\ &= f(0) + xf'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{3!} f'''(0) + \dots + \frac{x^n}{n!} f^{(n)}(0) + \dots \end{aligned}$$

- Taylor Series is a generalization of Maclaurin Series:

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!} f^{(k)}(a) \\ &= f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \dots + \frac{(x-a)^n}{n!} f^{(n)}(a) + \dots \end{aligned}$$

so we usually call Maclaurin series as Taylor series.

- With Taylor series, everything can be expressed as polynomial

- Example: Express  $e^x$  as Taylor series

$$\begin{aligned}e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} f^{(k)}(0) \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!} e^0 \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\end{aligned}$$

- Example: Express  $\sin x$  as Taylor series

- Example: Express  $\cos x$  as Taylor series

- Example: Express  $\frac{1}{x+1}$  as Taylor series