

Remedial Lesson 7: Application of Calculus II

Prepared by Adrian Sai-wah Tam
(swtam3@ie.cuhk.edu.hk)

October 10, 2005

1 FYI: Laplace Transform and Fourier Transform

- Laplace Transform:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \lim_{R \rightarrow \infty} \frac{1}{2\pi i} \int_{\beta-iR}^{\beta+iR} e^{st} F(s) ds$$

- Laplace transform is a function of functions, i.e.

- Input: a function in t
- Output: a function in s
- For nearly any function in t , we can find an unique corresponding function in s
- If we get the output, we can revert and get back the input (may differ by a constant)

- Use: Solving differential equations

- Table of Laplace Transform:

$f(t)$	$F(s)$	$f(t)$	$F(s)$
$f(t)$	$\int_0^{\infty} e^{-st} f(t) dt$	$\delta(t)$	1
$af(t) + bg(t)$	$aF(s) + bG(s)$	1 or $u(t)$	$\frac{1}{s}$
$e^{at} f(t)$	$F(s-a)$	$u(t-a)$	$\frac{1}{s} e^{-as}$
$f(t-a)u(t-a)$	$e^{-as} F(s)$	$\delta(t-a)$	e^{-as}
$f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right)$	te^{-at}	$\frac{1}{(s+a)^2}$
$f^{(n)}(t)$	$s^n F(s) - \sum_{k=0}^{n-1} s^{n-1-k} f^{(k)}(0)$	$t \sin \omega t$	$\frac{2\omega s}{(s^2 + \omega^2)^2}$
$\int_0^t f(\tau) d\tau$	$\frac{1}{s} F(s)$	$e^{at} \sin \omega t$	$\frac{\omega}{(s-a)^2 + \omega^2}$
$tf(t)$	$-F'(s)$	$e^{at} \cos \omega t$	$\frac{s-a}{(s-a)^2 + \omega^2}$
$\frac{1}{t} f(t)$	$\int_s^{\infty} F(\sigma) d\sigma$		
$f(t) * g(t)$	$F(s)G(s)$	$f'(t)$	$sF(s) - f(0)$
$f(0)$	$\lim_{s \rightarrow \infty} sF(s)$	$f''(t)$	$s^2 F(s) - sf(0) - f'(0)$
$f(\infty)$	$\lim_{s \rightarrow 0} sF(s)$	$tf'(t)$	$-F(s) - sF'(s)$
		$tf''(t)$	$-2sF(s) - s^2 F'(s) - f(0)$

- Fourier transform:

$$F(\omega) = \mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\omega x} dx$$

$$f(x) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)e^{i\omega x} d\omega$$

- Fourier transform is also a function of functions, i.e.

- Input: a function in time domain, t
- Output: a function in frequency domain, f

- Use: Frequency domain analysis

- Table of fourier transform

$f(t)$	$F(\omega)$	$f(t)$	$F(\omega)$
$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)e^{i\omega t} d\omega$	$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt$	$\delta(t)$	1
$af(t) + bg(t)$	$aF(\omega) + bG(\omega)$	$e^{i\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$
$e^{i\omega_0 t} f(t)$	$F(\omega - \omega_0)$	$u(t)$	$\pi\delta(\omega) + \frac{1}{i\omega}$
$f(t - t_0)$	$e^{-i\omega t_0} F(\omega)$	$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
$f(at)$	$\frac{1}{a} F\left(\frac{\omega}{a}\right)$	$\sin \omega_0 t$	$-i\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
$F(t)$	$2\pi f(-\omega)$	$u(t) \cos \omega_0 t$	$\frac{\pi}{2} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \frac{i\omega}{\omega_0^2 - \omega^2}$
$f^{(n)}(t)$	$(i\omega)^n F(\omega)$	$u(t) \sin \omega_0 t$	$\frac{-i\pi}{2} [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] + \frac{\omega^2}{\omega_0^2 - \omega^2}$
$(-it)^n f(t)$	$F^{(n)}(\omega)$	$u(t)e^{-at} \cos \omega_0 t$	$\frac{a + i\omega}{\omega_0^2 + (a + i\omega)^2}$
$\int_{-\infty}^t f(\tau) d\tau$	$\frac{1}{i\omega} F(\omega) + \pi F(0)\delta(\omega)$	$u(t)e^{-at} \sin \omega_0 t$	$\frac{\omega_0}{\omega_0^2 + (a + i\omega)^2}$
$f(t) * g(t)$	$\sqrt{2\pi} F(\omega)G(\omega)$	$u(t)e^{-at}$	$\frac{1}{a + i\omega}$
$f(t)g(t)$	$\frac{1}{2\pi} F(\omega) * G(\omega)$	$u(t)te^{-at}$	$\frac{1}{(a + i\omega)^2}$

2 Leibniz's Rule for Order- n Differentiation

- Given function $u(x) = f(x)g(x)$,

$$u'(x) = \frac{d}{dx}u(x) = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

$$u''(x) = \frac{d^2}{dx^2}u(x) = f(x)\frac{d^2}{dx^2}g(x) + 2\frac{d}{dx}f(x)\frac{d}{dx}g(x) + g(x)\frac{d^2}{dx^2}f(x)$$

$$\vdots$$

$$u^{(n)}(x) = \frac{d^n}{dx^n}u(x) = \sum_{k=0}^n \binom{n}{k} \frac{d^{n-k}}{dx^{n-k}}f(x)\frac{d^k}{dx^k}g(x)$$

this is called the Leibniz's rule. Which as the form similar to binomial theorem:

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

$$\frac{d^n}{dx^n} (f(x) \cdot g(x)) = \sum_{k=0}^n \binom{n}{k} f^{(n-k)}(x) g^{(k)}(x)$$

- Example:

$$\begin{aligned}
 y &= x^2 \sin x \\
 \frac{d}{dx} \sin x &= \sin\left(x + \frac{\pi}{2}\right) \\
 \frac{d^2}{dx^2} \sin x &= \sin\left(x + \frac{2\pi}{2}\right) \\
 &\vdots \\
 \frac{d^n}{dx^n} \sin x &= \sin\left(x + \frac{n\pi}{2}\right) \\
 \frac{d}{dx} x^2 &= 2x \\
 \frac{d^2}{dx^2} x^2 &= 2 \\
 &\vdots \\
 \frac{d^n}{dx^n} x^2 &= \begin{cases} 2x & : n = 1 \\ 2 & : n = 2 \\ 0 & : \text{otherwise} \end{cases} \\
 \therefore \frac{d^{80}}{dx^{80}} x^2 \sin x &= \binom{80}{0} x^2 \sin\left(x + \frac{80\pi}{2}\right) + \binom{80}{1} 2x \sin\left(x + \frac{79\pi}{2}\right) + \binom{80}{2} 2 \sin\left(x + \frac{78\pi}{2}\right) \\
 &= x^2 \sin x - 160x \cos x - 6320 \sin x
 \end{aligned}$$

Exercises

1. Find the n th derivative of $y = x^3 e^{ax}$, $n \geq 3$

$$\begin{aligned}
 \frac{d^n}{dx^n} (x^3 e^{ax}) &= \sum_{k=0}^n \binom{n}{k} \frac{d^{n-k}}{dx^{n-k}} e^{ax} \frac{d^k}{dx^k} x^3 \\
 &= \binom{n}{0} x^3 a^n e^{ax} + \binom{n}{1} 3x^2 a^{n-1} e^{ax} + \binom{n}{2} 6xa^{n-2} e^{ax} + \binom{n}{3} 6a^{n-3} e^{ax} \\
 &= x^3 a^n e^{ax} + n3x^2 a^{n-1} e^{ax} + \frac{n(n-1)}{2!} 6xa^{n-2} e^{ax} + \frac{n(n-1)(n-2)}{3!} 6a^{n-3} e^{ax} \\
 &= x^3 a^n e^{ax} + n3x^2 a^{n-1} e^{ax} + 3n(n-1)xa^{n-2} e^{ax} + n(n-1)(n-2)a^{n-3} e^{ax} \\
 &= e^{ax} [x^3 a^n + n3x^2 a^{n-1} + 3n(n-1)xa^{n-2} + n(n-1)(n-2)a^{n-3}]
 \end{aligned}$$

2. Find the n th derivative of $y = 2^x \ln x$

$$\begin{aligned}
 \frac{d^n}{dx^n} 2^x \ln x &= \sum_{k=0}^n \binom{n}{k} \frac{d^{n-k}}{dx^{n-k}} 2^x \frac{d^k}{dx^k} \ln x \\
 &= \sum_{k=0}^n \binom{n}{k} \frac{d^{n-k}}{dx^{n-k}} e^{x \ln 2} \frac{d^k}{dx^k} \ln x \\
 &= e^{x \ln 2} (\ln 2)^n \ln x + \sum_{k=1}^n \binom{n}{k} \frac{d^{n-k}}{dx^{n-k}} e^{x \ln 2} \frac{d^k}{dx^k} \ln x \\
 &= e^{x \ln 2} (\ln 2)^n \ln x + \sum_{k=1}^n \binom{n}{k} \left[e^{x \ln 2} (\ln 2)^{n-k} \right] \cdot \left[(-1)^{k-1} \frac{(k-1)!}{x^k} \right] \\
 &= 2^x \left[(\ln 2)^n \ln x + \sum_{k=1}^n \binom{n}{k} (-1)^{k-1} (\ln 2)^{n-k} \frac{(k-1)!}{x^k} \right]
 \end{aligned}$$

3 Finding Local Extrema

- Given a curve $y = f(x)$, if the point (x', y') is the maximum or minimum of the curve, the slope at that point must be zero, i.e.

$$\left. \frac{d}{dx} f(x) \right|_{x=x'} = 0$$

- If it is maximum, the slope of $y = f(x)$ should be decreasing (from positive slope, to zero, to negative), but if it is minimum, the slope of $y = f(x)$ should be increasing (from negative slope, to zero, to positive).
- Point of inflexion is the point (x', y') that gives a zero slope, but it is neither maximum nor minimum
- Example: Find the maximum and minimum values of $f(x) = (x+2)^2(x-1)^3$

$$f(x) = (x+2)^2(x-1)^3$$

$$f'(x) = 2(x+2)(x-1)^3 + 3(x+2)^2(x-1)^2$$

$$= (x+2)(x-1)^2(5x+4)$$

$$\therefore f'(x) = 0 \implies x = -2, 1, -\frac{4}{5}$$

$$x < -2 \implies f'(x) > 0$$

$$-2 < x < -\frac{4}{5} \implies f'(x) < 0$$

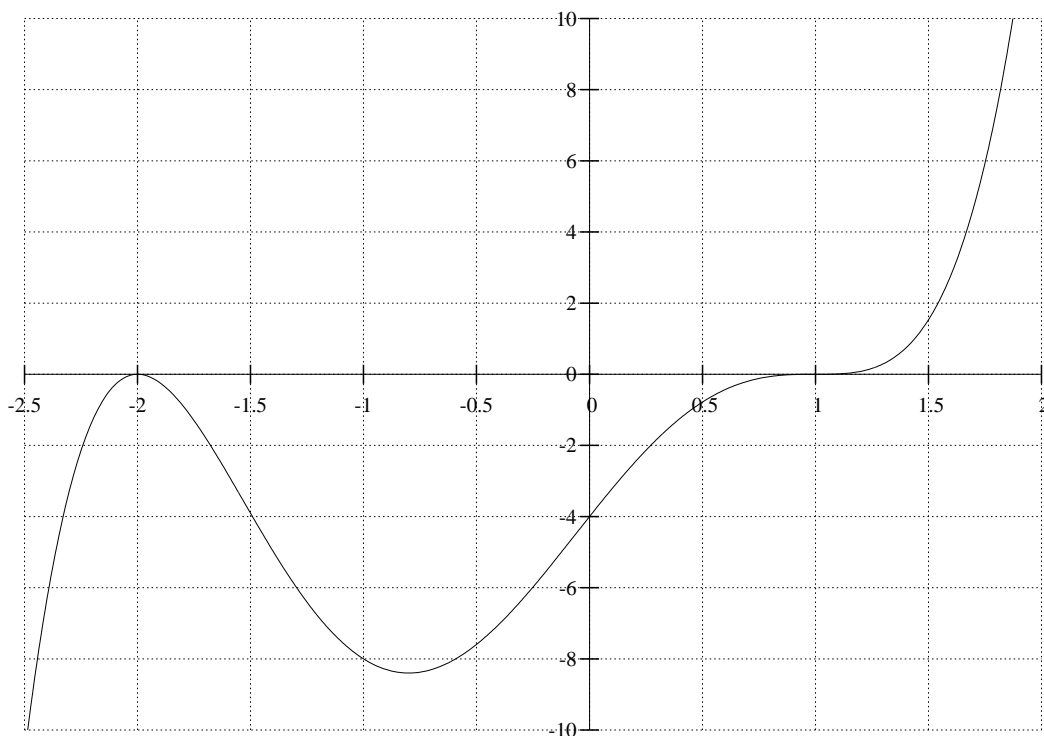
$$-\frac{4}{5} < x < 1 \implies f'(x) > 0$$

$$x > 1 \implies f'(x) > 0$$

$$\therefore \text{minimum at } x = -\frac{4}{5} \text{ at } \left(-\frac{4}{5}, -\frac{26244}{3125}\right)$$

$$\text{maximum at } x = -2 \text{ at } (-2, 0)$$

$$\text{inflexion at } x = 1 \text{ at } (1, 0)$$



Exercises

1. Find the maximum and minimum values of $f(x) = \sin^3 x + \cos^3 x$

$$f(x) = \sin^3 x + \cos^3 x$$

$$f'(x) = 3\sin^2 x \cos x - 3\cos^2 x \sin x$$

$$= 3\sin x \cos x (\sin x - \cos x)$$

$$f'(x) = 0 \implies x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, 2\pi, \dots$$

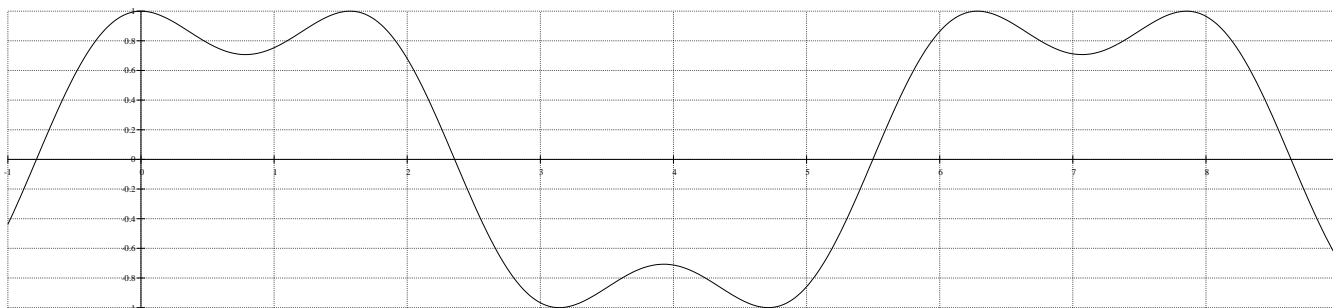
By checking, minima at: $x = \frac{\pi}{4}, \pi, \frac{3\pi}{2}$

maxima at: $x = 0, \frac{\pi}{2}, \frac{5\pi}{4}, 2\pi$

with the minima points: $(\frac{\pi}{4}, \frac{\sqrt{2}}{2}) (\pi, -1) (\frac{3\pi}{2}, -1)$

maxima points: $(0, 1) (\frac{\pi}{2}, 1) (\frac{5\pi}{4}, 1) (2\pi, 1)$

and periodic with period 2π



2. Find the values of x of the function $f(x) = \frac{x^2 - 5x + 6}{x^2 + 1}$

$$f(x) = \frac{x^2 - 5x + 6}{x^2 + 1}$$

$$f'(x) = \frac{(x^2 + 1)(2x - 5) - (x^2 - 5x + 6)(2x)}{(x^2 + 1)^2}$$

$$= \frac{5x^2 - 10x - 5}{(x^2 + 1)^2}$$

$$= \frac{5(x^2 - 2x - 1)}{(x^2 + 1)^2}$$

$$f'(x) = 0 \implies x = 1 \pm \sqrt{2}$$

By checking, minima at: $x = 1 + \sqrt{2}$

maxima at: $x = 1 - \sqrt{2}$

with the minima points: $(1 + \sqrt{2}, \frac{4 - 3\sqrt{2}}{4 + 2\sqrt{2}})$

maxima points: $(1 - \sqrt{2}, \frac{4 + 3\sqrt{2}}{4 - 2\sqrt{2}})$

