ERG2011A Tutorial 2: Vector Differentiation

Prepared by Adrian Sai-wah TAM (swtam3@ie.cuhk.edu.hk)

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1 Vector-valued Functions

- Summary from Tutorial 1:

$$\text{Scalar-valued function}: \begin{cases} f(x) = \sin(x) \\ f(x) = \sqrt{x} \\ f(\vec{x}) = |\vec{x}| = \sqrt{\vec{x}\cdot\vec{x}} \end{cases}$$

$$(\qquad\qquad)\text{-valued function}: \begin{cases} \vec{a}(\vec{F}) = \vec{F}/m \\ \vec{c}(\vec{x},\vec{y}) = \vec{x}\times\vec{y} \\ \vec{v}(t) = 4t\vec{i} + \frac{2}{t}\vec{j} \end{cases}$$

- Let's think of simply 3D Cartesian coordinates ($\mathbb{R}^3$) in this course:

$$\mathbf{v}(t) = [v_x(t), v_y(t), v_z(t)]$$

- Differentiation of vector:

$$\frac{d}{dt}\mathbf{v}(t) = \underline{\qquad\qquad\qquad\qquad}$$

  - But be careful that: Vector minus another vector is a (   ), hence the derivative has also direction and magnitude
  - Physical meaning of vector derivative: Rate of change of a vector, i.e. magnitude and direction

- Properties of vector differentiation: (just remember this)

	Vector Differentiation	Scalar Differentiation
Constant multiplication	$\frac{d}{dt}c\mathbf{v}(t) = c\frac{d}{dt}\mathbf{v}(t)$	$\frac{d}{dx}cf(x) = c\frac{d}{dx}f(x)$
Addition	$\frac{d}{dt}[\mathbf{u}(t)+\mathbf{v}(t)] = \frac{d\mathbf{u}(t)}{dt} + \frac{d\mathbf{v}(t)}{dt}$	$\frac{d}{dx}[f(x)+g(x)] = \frac{df(x)}{dx} + \frac{dg(x)}{dx}$
Chain rules	$\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}(t)\cdot\frac{d\mathbf{v}(t)}{dt} + \frac{d\mathbf{u}(t)}{dt}\cdot\mathbf{v}(t)$ $\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}(t)\times\frac{d\mathbf{v}(t)}{dt} + \frac{d\mathbf{u}(t)}{dt}\times\mathbf{v}(t)$	$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{dg(x)}{dx} + \frac{df(x)}{dx}g(x)$

  - and hence we have $\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)\times\mathbf{w}(t)] = \frac{d\mathbf{u}(t)}{dt}\cdot\mathbf{v}(t)\times\mathbf{w}(t) + \mathbf{u}(t)\cdot\frac{d\mathbf{v}(t)}{dt}\times\mathbf{w}(t) + \mathbf{u}(t)\cdot\mathbf{v}(t)\times\frac{d\mathbf{w}(t)}{dt}$

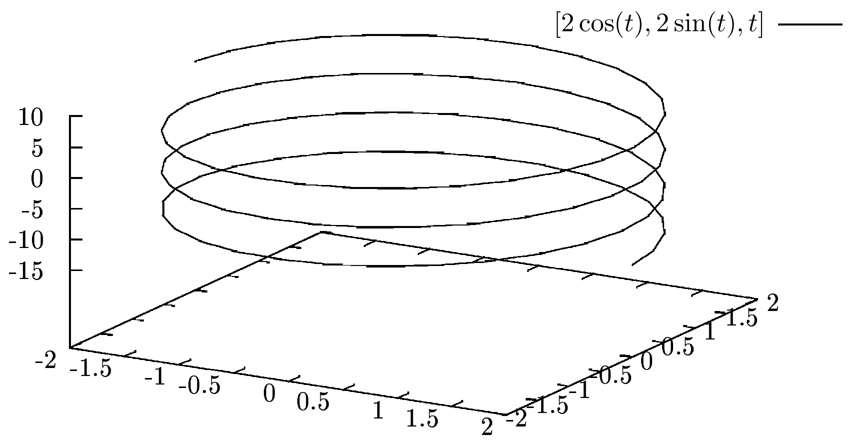

Figure 1: Circular helix

- Because we are in 3D, we may using this notation to express the derivative components:

$$\begin{aligned} \mathbf{v}'(t) &= [v_x(t), v_y(t), v_z(t)]' \\ &= \frac{d}{dt}v_x(t)\mathbf{i} + \frac{d}{dt}v_y(t)\mathbf{j} + \frac{d}{dt}v_z(t)\mathbf{k} \end{aligned}$$

2 Use of Vector Derivative: Curves & Tangents

- Using vector to replace parametric notation to represent a curve:

$$\mathbf{r}(t) = [x(t), y(t), z(t)] = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

  - Example: Circular helix, $\mathbf{r}(t) = [a\cos(t),\ a\sin(t),\ ct]$. Figure 1 is $[2\cos(t),\ 2\sin(t),\ t]$.
  - Example: Straight line, $\mathbf{s}(t) = \mathbf{a} + t\mathbf{b}$

- Question: How to find the tangent to the helix $\mathbf{r}(t)$ at point $(\frac{a\sqrt{3}}{2}, \frac{a}{2}, \frac{13\pi a}{6})$?

  - Answer: Let the tangent be $\mathbf{s}(t)$, then it has the form

$$\mathbf{s}(t) = \qquad + \qquad + \qquad + t\mathbf{b}$$

    where $\mathbf{b}$ is some vector parallel to the tangent

  - Since tangent is defined as the ( ), then we know that the vector along the tangent can be written as:

hence we have:

$$\begin{aligned} \mathbf{s}(t) &= \left(\frac{a\sqrt{3}}{2}, \frac{a}{2}, \frac{13\pi a}{6}\right) + \\ &= \left(\frac{a\sqrt{3}}{2} - at\sin(\frac{\pi}{6})\right)\mathbf{i} + \left(\frac{a}{2} + at\cos(\frac{\pi}{6})\right)\mathbf{j} + \left(\frac{13\pi a}{6} + ct\right)\mathbf{k} \end{aligned}$$

- Conclusion:

  Curve $\mathbf{r}(t)$ will have its tangent at the point $\mathbf{r}(\tau)$ in the form $\mathbf{s}(t) = \mathbf{r}(\tau) + t\mathbf{r}'(\tau)$

  - If you like, you can find the *unit tangent vector*, $\mathbf{u} = \frac{1}{|\mathbf{r}'(\tau)|}\mathbf{r}'(\tau)$

3 Use of Vector Derivative: Length of Curve

- Remember: Tangent is the limit of chord
- Hence, magnitude of tangent is related to the limiting chord length, i.e. curve length
- Actually, we can have the following formula:

$$\begin{aligned} \ell &= \int_a^b | \qquad\qquad |dt = \text{Curve length from the point } \mathbf{r}(a) \text{ to } \mathbf{r}(b) \\ &= \int_a^b \sqrt{\mathbf{r}'(t)\cdot\mathbf{r}'(t)}dt \end{aligned}$$

- Example: $\mathbf{r}(t) = a\cos(t)\mathbf{i} + a\sin(t)\mathbf{j} + ct\mathbf{k}$, find the curve length from $(a, 0, 0)$ to $(a, 0, 2\pi c)$

  1. We want to represent the points in t,

$$\begin{cases} a\cos(t) &= a \\ a\sin(t) &= 0 \\ ct &= 0 \end{cases} \implies t =$$

$$\begin{cases} a\cos(t) &= a \\ a\sin(t) &= 0 \\ ct &= 2\pi c \end{cases} \implies t =$$

  2. The derivative of $\mathbf{r}(t)$:

$$\mathbf{r}'(t) =$$

  3. So, the curve length is:

$$\begin{aligned} \ell &= \int \sqrt{\mathbf{r}'(t)\cdot\mathbf{r}'(t)}dt \\ \\ &= \\ \\ &= \end{aligned}$$