ERG2011A Tutorial 4: More Vector Integration

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1 Green's Theorem in the Plane

- Line integral: $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$
  - Summing up all the vectors from function $\mathbf{F}(\mathbf{r})$ where $\mathbf{r}$ is a vector parameter (e.g. a point in space) supplied to $\mathbf{F}$, sweeping along curve C
  - If curve C is a closed loop, we may write $\oint_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$
- Double integral can be transformed into line integral:

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \oint_C (F_1 dx + F_2 dy)$$

  - Known as *Green's Theorem*
  - R is a closed bounded region in the xy-plane and its boundary is C
  - C consists of finite smooth curves
  - Functions $F_1(x, y)$ and $F_2(x, y)$ are continuous and differentiable everywhere in R as required
- Alternative form of Green's Theorem:

$$\iint_R (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} dxdy = \oint_C \mathbf{F} \cdot d\mathbf{r} \quad \text{where } \mathbf{F} = F_1 \mathbf{i} + F_2 \mathbf{j}$$

  - Proof of Green's Theorem see book pp.486-488.
- Remember a fact at this moment:
  - Counterclockwise is ( )
  - Clockwise is ( )

1.1 Application of Green's Theorem

- Area of plane by using line integral: $A = \frac{1}{2} \oint_C (xdy - ydx)$
  - Example: Problem Set 9.4, Question 12
    * Cycloid (in vector notation): $\mathbf{r} = a(t - \sin t)\mathbf{i} + a(1 - \cos t)\mathbf{j}$, for $0 \le t \le 2\pi$
    * Parametric form: $\begin{cases} x & = \\ y & = \end{cases}$
    * From the plot, we can see that the direction of *counterclockwise* is for t from to

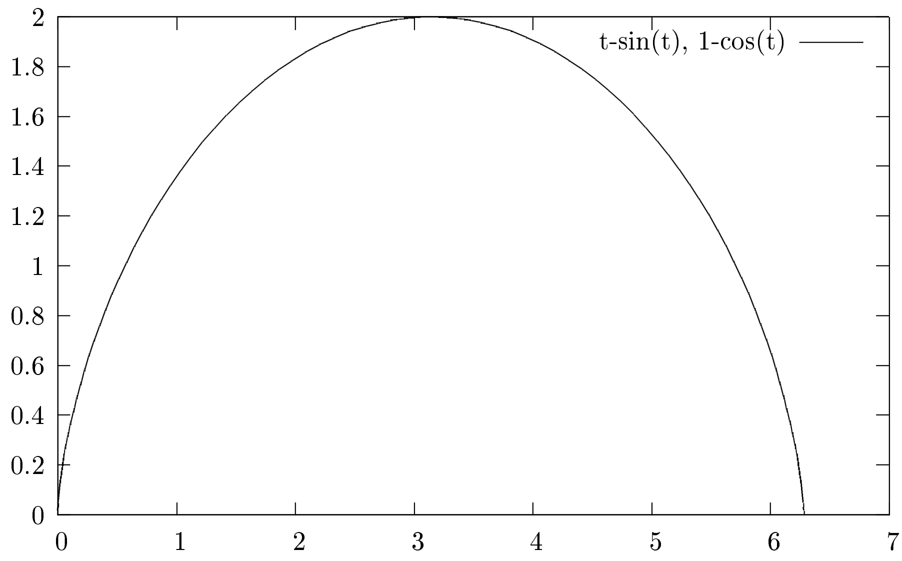

Figure 1: Problem Set 9.4, Question 12

* Area:

$$
\begin{aligned}
A &= \frac{1}{2}\oint_C (xdy - ydx) \\
&= \frac{1}{2}\int_{2\pi}^{0} \qquad dt \\
&= \frac{1}{2}\int_{2\pi}^{0} \left[a^2(t\sin t - \sin^2 t) - \qquad \right] dt \\
&= \frac{a^2}{2}\int_{2\pi}^{0} \left[t\sin t - \sin^2 t - 1 + 2\cos t - \cos^2 t\right] dt \\
&= \frac{a^2}{2}\int_{2\pi}^{0} [\qquad] dt \\
&= \frac{a^2}{2}\Big[\sin t - t\cos t + 2\sin t - 2t\Big]_{2\pi}^{0} =
\end{aligned}
$$

- Polar version: $A = \frac{1}{2}\oint_C r^2 d\theta$
  - Example: Problem Set 9.4, Question 13
    * Limaçon (in polar equation): $r = 1 + 2\cos\theta$, for $0 \le \theta \le \pi/2$
    * Because in polar, the counterclockwise is well known, i.e. for θ from to
    * Area is therefore:

$$
\begin{aligned}
A &= \frac{1}{2}\oint_C r^2 d\theta \\
&= \frac{1}{2}\int_{0}^{\pi/2} (1 + 2\cos\theta)^2 d\theta \\
&= \frac{1}{2}\int_{0}^{\pi/2} \qquad d\theta \\
&= \frac{1}{2}\int_{0}^{\pi/2} \left[1 + 4\cos\theta + 4\left(\frac{1+\cos 2\theta}{2}\right)\right] d\theta \\
&= \frac{1}{2}\Big[3\theta + 4\sin\theta + \sin 2\theta\Big]_{0}^{\pi/2} =
\end{aligned}
$$

2 Surface Integrals

2.1 Tangents of surface

- Surface in the xyz-space: $z = f(x, y)$ or $g(x, y, z) = 0$
  Parametric form of surface: $\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$
  - Note that curve in space is $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, which has only (   ) parameter, not two
- Example surface: A sphere: $\mathbf{r}(u, v) = a\cos u \cos v\mathbf{i} + a\cos v \sin u\mathbf{j} + a\sin v\mathbf{k}$
- For a curve, the tangent is a line as the limit of the chord
  For a surface, the tangent is a plane containing all the tangent of the curves on that surface
  - A curve on the surface can be defined by relating parameters u and v:

$$\tilde{\mathbf{r}}(t) = \mathbf{r}(u(t), v(t))$$

  - Tangents of this curve is therefore:

$$\tilde{\mathbf{r}}'(t) = \frac{d\tilde{\mathbf{r}}}{dt} = \frac{\partial \mathbf{r}}{\partial u}u' + \frac{\partial \mathbf{r}}{\partial v}v' = \mathbf{r}_u u' + \mathbf{r}_v v'$$

  - Therefore, the tangent plane is a plane spanned by   , i.e. $h\frac{\partial \mathbf{r}}{\partial u} + k\frac{\partial \mathbf{r}}{\partial v}$.
- Because the tangent plane is spanned by $\mathbf{r}_u$ and $\mathbf{r}_v$, the normal on that plane is in the direction $\mathbf{N} =$(   ).
  - *Unit* normal vector of a surface is defined to be: $\mathbf{n} = \frac{1}{|\mathbf{r}_u \times \mathbf{r}_v|}\mathbf{r}_u \times \mathbf{r}_v$
  - If the surface S is represented by $g(x, y, z) = 0$, in addition, we can get the unit normal vector by: $\mathbf{n} = \frac{1}{|\operatorname{grad} g|}\operatorname{grad} g$
- **Remember**: The actual tangent plane and unit normal vector depends on the (   ) of surface
- Example: Problem Set 9.5, Question 9
  Find the normal vector of the elliptic cylinder: $\mathbf{r}(u, v) = [a\cos v,\ b\sin v,\ u]$
  - $\frac{\partial \mathbf{r}}{\partial u} =$
  - $\frac{\partial \mathbf{r}}{\partial v} = -a\sin v\mathbf{i} + b\cos v\mathbf{j}$
  - Normal vector $\mathbf{N} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ -a\sin v & b\cos v & 0 \end{vmatrix} =$

2.2 Flux Integral / Surface Integral

- The flux (mass of fluid per unit time) across a surface is given by the surface integral,

$$\iint_S \mathbf{F}\cdot\mathbf{n}dA = \iint_R \mathbf{F}[\mathbf{r}(u,v)]\cdot\mathbf{N}(u,v)dudv$$

  - S in domain A is identical to R in the uv-plane
  - In calculation, we may write:

$$\begin{aligned}\mathbf{F} &= F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k}\\ \mathbf{n} &= \cos\alpha\mathbf{i}+\cos\beta\mathbf{j}+\cos\gamma\mathbf{k}\\ N &= N_1\mathbf{i}+N_2\mathbf{j}+N_3\mathbf{k}\end{aligned}$$

    then we can have:

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}dA &= \iint_S (F_1\cos\alpha+F_2\cos\beta+F_3\cos\gamma)dA\\ \iint_S \mathbf{F}\cdot\mathbf{n}dA &= \iint_R (F_1N_1+F_2N_2+F_3N_3)dudv\\ \iint_S \mathbf{F}\cdot\mathbf{n}dA &= \iint_S (F_1dydz+F_2dzdx+F_3dxdy)\end{aligned}$$

    They are all equivalent. Read book pp.496-500 for the derivation of them.
  - Example: Problem Set 9.6, Question 9
    $\mathbf{F}=[x,\ y,\ z]$, $S:\ \mathbf{r}=[u\cos v,\ u\sin v,\ u^2]$ where $0\le u\le 4$ and $-\pi\le v\le\pi$. Find $\iint_S \mathbf{F}\cdot\mathbf{n}dA$
    * From $\mathbf{r}$, we have: $\dfrac{\partial\mathbf{r}}{\partial u} =$, $\dfrac{\partial\mathbf{r}}{\partial v} =$
    * Therefore, $\dfrac{\partial\mathbf{r}}{\partial u}\times\dfrac{\partial\mathbf{r}}{\partial v} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \cos v & \sin v & 2u\\ -u\sin v & u\cos v & 0\end{vmatrix} = -2u^2\cos v\mathbf{i} - 2u^2\sin v\mathbf{j} + u\mathbf{k}$
    * Thus,

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}dA &= \iint_S \mathbf{F}(\mathbf{r}(u,v))\cdot\mathbf{N}dudv\\ &= \int_{-\pi}^{\pi}\int_0^4 \left[u\cos v\,(\qquad) + u\sin v\,(\qquad) + u^2\,(\quad)\right] dudv\\ &= \int_{-\pi}^{\pi}\int_0^4 \left[-u^3\right] dudv\\ &= \int_{-\pi}^{\pi}(-\frac{1}{4}4^4)dv\\ &= \int_{-\pi}^{\pi}(-64)dv\\ &= -64(2\pi)\\ &=\end{aligned}$$

- Please be careful that the normal vector is directional, hence the surface integral can be ( )
  - To calculate surface integral without regard to ( ), we have another type of surface integral:

$$\iint_S G(\mathbf{r})dA = \iint_R G(\mathbf{r}(u,v))|\mathbf{N}(u,v)|dudv$$

$$\iint_S G(\mathbf{r})dA = \iint_{R'} G(x,y,f(x,y))\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}dxdy$$

which is used in the calculation of moment of inertia

  - Example: Problem Set 9.6, Question 15
    $G = (1+9xz)^{3/2}$, $S:\ \mathbf{r} = [u,\ v,\ u^3]$ where $0 \le u \le 1$ and $-2 \le v \le 2$. Find $\iint_S G(\mathbf{r})dA$
    * We have, $\begin{cases} x &= \\ y &= \\ z &= \end{cases}$
    * Therefore, $\begin{cases} dx &= du \\ dy &= dv \\ dz &= 3u^2du \end{cases}$
    * Thus,

$$\begin{aligned}
\iint_S G(\mathbf{r})dA &= \iint_{R'} G(x,y,z)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}dxdy \\
&= \int_{-2}^{2}\int_0^1 (\qquad\qquad)^{3/2}(\qquad\qquad)^{1/2}dudv \\
&= \int_{-2}^{2}\left(\int_0^1 \left(1+9u^4\right)^2 du\right)dv \\
&= \int_{-2}^{2}\left(\int_0^1 \left(1+18u^4+81u^8\right)du\right)dv \\
&= \int_{-2}^{2}\Big[\qquad\qquad\Big]_0^1 dv \\
&= \int_{-2}^{2}\frac{68}{5}dv \\
&=
\end{aligned}$$