ERG2011A Tutorial 5: Two Big Theorems in Vector Analysis

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1 Triple Integrals

- We have learnt double integral and surface integral:

$$\begin{aligned} \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy &= \oint_C (F_1 dx + F_2 dy) \\ \iint_S \mathbf{F} \cdot \mathbf{n} dA &= \iint_R \mathbf{F}[\mathbf{r}(u,v)] \cdot \mathbf{N}(u,v) dudv \end{aligned}$$

- Triple integral is the same concept: summing up the values over a ( ):

$$\iiint_R f(x,y,z)dxdydz = \iiint_R f(x,y,z)dV$$

  - Example: Density $\times$ Volume $=$ Mass, therefore for an item with uneven density, $\iiint_R(\text{density})dV =$
  - Calculating triple integral is the same as calculating double integral, i.e. transform it into ( ) integral first

- Example: Problem Set 9.7 Question 2
  Find the total mass of a mass distribution of density σ in a region T in space, where $\sigma = e^{-x-y-z}$ and $T: 0 \le x \le 1-y$, $0 \le y \le 1$, $0 \le z \le 2$
  - Density function: $f(x,y,z) = e^{-x-y-z}$
  - Boundary of the region: T
  - Total mass $=$ Integral $=$

$$\begin{aligned} \iiint_T \sigma dV &= \int_{(\ )}^{(\ )} \int_{(\ )}^{(\ )} \int_{(\ )}^{(\ )} e^{-x-y-z} dxdydz \\ &= \int_0^2 \int_0^1 [-e^{-x-y-z}]_0^{1-y} dydz \\ &= \int_0^2 \int_0^1 (-e^{-1-z} + e^{-y-z}) dydz \\ &= \int_0^2 [\qquad\qquad]_0^1 dz \\ &= \int_0^2 (-2e^{-1-z} + e^{-z}) dz \\ &= [\qquad\qquad]_0^2 \\ &= 2e^{-3} - e^{-2} - 2e^{-1} + 1 \end{aligned}$$

2 Gauss' Divergence Theorem

- GDT is analogous to Green's theorem in 3D
  - Triple integral of divergence can be transformed into the surface integral:

- Variations of Divergence Theorem:
  1. $\iiint_T \nabla \cdot \mathbf{F} dV = \iint_S \mathbf{F} \cdot \mathbf{n} dA$
  2. $\iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dxdydz = \iint_S (F_1 dydz + F_2 dzdx + F_3 dxdy)$
- Occationally, we use the GDT in the ( ) way, i.e. given the integral at the right hand side and transform it into integral in left hand side.
- Example: Problem Set 9.7 Question 14
  Evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n} dA$ by the divergence theorem for $\mathbf{F} = [e^x, e^y, e^z]$, S is the surface of the cube $|x| \le 1$, $|y| \le 1$, $|z| \le 1$.
  - Surface are ( ) flat planes that parallel to ( )-plane, ( )-plane and ( )-plane.
  - Normal unit vectors are therefore, depend on which of the six planes, one of the following:

  - Instead of summing six integrals to make up the whole surface S, we use the GDT, hence:

$$
\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n} dA &= \iiint_T \nabla \cdot \mathbf{F} dV \\
&= \int_{(\ )}^{(\ )} \int_{(\ )}^{(\ )} \int_{(\ )}^{(\ )} (e^x + e^y + e^z) dxdydz \\
&= \int_{-1}^{1} \int_{-1}^{1} [e^x + xe^y + xe^z]_{-1}^{1} dydz \\
&= \int_{-1}^{1} \int_{-1}^{1} \left(e - e^{-1} + e^y + e^y + e^z + e^z\right) dydz \\
&= \int_{-1}^{1} \left[(e - e^{-1})y + 2e^y + 2ye^z\right]_{-1}^{1} dz \\
&= \int_{-1}^{1} \left(4e - 4e^{-1} + 4e^z\right) dz \\
&= \left[(4e - 4e^{-1})z + 4e^z\right]_{-1}^{1} \\
&= 12e - 12e^{-1}
\end{aligned}
$$

- As you see from the triple integral's example, we may use triple integral to find the mass.
  But if the density is uniform unity, the mass is identical to the ( )

- Example: Problem Set 9.8 Question 11
  Show that a region T with boundary surface S has the volume:
  $V = \iint_S xdydz = \iint_S ydzdx = \iint_S zdxdy = \frac{1}{3}\iint_S (xdydz + ydzdx + zdxdy)$
  - Because volume can be calculated by using the triple integral: $V = \iiint_{S^\star} (\qquad) dxdydz$ where $S^\star$ means the region bounded by the surface, by GDT, we have:

$$\begin{aligned} V &= \iiint_{S^\star} (1) dxdydz \\ &= \iiint_{S^\star} \frac{1}{3} (\nabla \cdot [x,\, y,\, z])\, dxdydz \\ &= \frac{1}{3} \iiint_{S^\star} \nabla \cdot [x,\, y,\, z] dxdydz \\ &= \frac{1}{3} \iint_S (xdydz + ydzdx + zdxdy) \end{aligned}$$

  - Alternatively, we also found that: $\nabla \cdot [x,\, 0,\, 0] = \qquad$, $\nabla \cdot [0,\, y,\, 0] = \qquad$ and $\nabla \cdot [0,\, 0,\, z] = \qquad$. Repeating the above derivation can then show the result.
- Hence you can see, find *any* vector function $\mathbf{F}$ that fit the problem, then you can use GDT.

3 Stroke's Theorem

3.1 Meaning of curl

- Remember that, for a particle rotation along an axis, such that the locus of rotation has radius $\mathbf{r}$, rotating with ($\qquad\qquad\qquad\qquad$) ω and instantaneous velocity $\mathbf{v}$ is related by:

$$\omega \times \mathbf{r} = \mathbf{v}$$

- You can verify this: $\nabla \times \mathbf{v} = \frac{1}{2}\omega$
- Thus "curl" means angular velocity of a vector field
  - If "curl" is zero, it is ($\qquad\qquad\qquad$)
  - Measuring the angular velocity can give a sense on the value of curl
- Example: If $\mathbf{v}$ is the flow of water and I put a paddle wheel on the water, will it rotate?
  1. Draw a plane surface to contain the wheel
  2. Draw a closed loop on the surface
  3. Sum up all the parallel-to-surface component of vectors $\mathbf{v}$ along the curve (line integral)
  4. The summation is not zero, the wheel will rotate
- But alternatively, we can also think of this:
  1. The surface is full of tiny gears
  2. There is a gear at the axis of the paddle wheel
  3. Gears may rotate clockwisely or counterclockwisely, fast or slow
     which can be represented by $(\nabla \times \mathbf{v}) \cdot \mathbf{n}$ where $\mathbf{n}$ is the normal unit vector to the surface
  4. If the rotation of all the gears are balanced, i.e. $\sum(\nabla \times \mathbf{v}) \cdot \mathbf{n} = 0$, the gear representing the paddle wheel will not rotate

3.2 Stroke's theorem

- Stroke's theorem:

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dxdy = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

- It is a generalization of Green's theorem to the 3D space

- Example: Problem Set 9.9 Question 2
  Integrate the surface integral $\iint_S (\text{curl}\,\mathbf{F}) \cdot \mathbf{n} dA$ directly. Then check the result by integrating the corresponding line integral by Stoke's theorem: $\iint_S (\text{curl}\,\mathbf{F}) \cdot \mathbf{n} dA = \oint_C \mathbf{F} \cdot \mathbf{r}'(s) ds$. Where:$\mathbf{F} = [y^2, -x^2, 0]$, S is the circular semidisk $x^2 + y^2 \leq 4$, $y \geq 0$, $z = 0$.

  - Direct integration:

$$\begin{aligned} \text{curl}\,\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & -x^2 & 0 \end{vmatrix} \\ &= (\qquad\qquad\qquad\qquad)\mathbf{k} \end{aligned}$$

$$\mathbf{n} = \mathbf{k}$$

$$\begin{aligned} \text{Hence:} \quad (\nabla \times \mathbf{F}) \cdot \mathbf{n} &= -2x - 2y \\ \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dA &= \int_0^2 \int_{-\sqrt{\qquad}}^{\sqrt{\qquad}} (-2x - 2y) dxdy \\ &= \int_0^2 \left[-x^2 - 2xy\right]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dy \\ &= \int_0^2 \left(-4y\sqrt{4-y^2}\right) dy \\ &= -2\int_0^4 \sqrt{4-u} du \\ &= -2\left[-\frac{(4-u)^{3/2}}{3/2}\right]_0^4 \\ &= -\frac{32}{3} \end{aligned}$$

  - Stroke's theorem:

    * The perimeter of the circular semidisk in parametric form:

$$x^2 + y^2 = 4 \qquad \Longrightarrow \qquad \begin{cases} x = \\ y = \end{cases}$$

with $0 \leq t \leq \pi$

* Line integral on curve part of the perimeter:

$$\begin{aligned}
\oint_C \mathbf{F}\cdot\mathbf{r}'(s)ds &= \int_0^{\pi} [y^2,\ -x^2,\ 0]\cdot[-2\sin t,\ 2\cos t,\ 0]dt \\
&= \int_0^{\pi} (-8\sin^3 t - 8\cos^3 t)dt \\
&= -2\int_0^{\pi} (3\sin t - \sin 3t + \cos 3t + 3\cos t)dt \\
&= -2\left[-3\cos t + \frac{1}{3}\cos 3t + \frac{1}{3}\sin 3t + 3\sin t\right]_0^{\pi} \\
&= -2\left(3 - \frac{1}{3} + 3 - \frac{1}{3}\right) \\
&= -\frac{32}{3}
\end{aligned}$$

* Line integral on the straight line part of the perimeter: $(-2, 0) \to (2, 0)$

$$\mathbf{r}(t) = t\mathbf{i} \quad \text{with } t \text{ from } -2 \text{ to } 2$$

Therefore:

$$\mathbf{r}'(t) = [1, 0, 0]$$

$$\begin{aligned}
\oint_C \mathbf{F}\cdot\mathbf{r}'(s)ds &= \int_{-2}^{2} [0, -t^2, 0]\cdot[1, 0, 0]dt \\
&= \int_{-2}^{2} (0)dt \\
&= 0
\end{aligned}$$

* Which shows that the result from Stroke's theorem and from direct integration are the same.