ERG2011A Tutorial 6: Differential Equations

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1 Equations Review

- These are two equations:

$$\begin{aligned} x^2 + 2x - 3 &= 0 \\ \sin x + \cos x &= \sqrt{2}/2 \end{aligned}$$

  - Equations have (unknowns)
  - Solving an equation means defining the unknown (explicitly)

$$\begin{aligned} \sin x + \cos x &= \sqrt{2}/2 \\ \sqrt{2}\sin(x + \pi/4) &= \sqrt{2}/2 \\ \sin(x + \pi/4) &= 1/2 \\ x + \pi/4 &= \sin^{-1}(1/2) \\ x &= n\pi + (-1)^n \frac{\pi}{6} - \frac{\pi}{4} \end{aligned}$$

  - Note: We solve for $\sin x + \cos x = \sqrt{2}/2$ and we have used $\sin^{-1}(\frac{1}{2})$, i.e. the (inverse) function of sine.

2 Integraion and Differential Equation

- (Indefinite) integral is defined as the (inverse) function of differentiation
  - Example: $f(x) = \sin x + \cos x$, then $\frac{d}{dx} f(x) = \cos x - \sin x$. Therefore,

$$\int (\cos x - \sin x) dx = \sin x + \cos x + C$$

    for some constant of integration C.

- From the differential equation's point of view, $\frac{d}{dx} f(x)$ is just a (function) of x. And its inverse function is (integration).

3 Differential Equations

- Differential equation may not be solvable. But some are solvable, which is our scope of study.

- Example: Mid-term exam, question 7: $y' = -y$

$$\begin{aligned} \frac{dy}{dx} &= -y \\ (-\frac{dy}{y} &= dx) \\ -\int \frac{1}{y} dy &= \int dx \\ -\ln y &= x + (C) \qquad \leftarrow \text{we add the constant of integration as soon as possible} \\ y &= e^{-x-C} = e^{-x}e^{-C} \\ &= C'e^{-x} \end{aligned}$$

for some constant C'. But since the question told us that $y(0) = 3$, we have:

$$\begin{aligned} C'e^{(-0)} &= 3 \\ C'(1) &= 3 \\ \therefore \quad y &= 3e^{-x} \end{aligned}$$

- In the above equation, we call it (first-order, linear, homogeneous) differential equation
  - First-order: No derivatives of over degree-1
  - Linear: We don't have $\sin y'$ or yy' or $(y')^2$
  - Homogeneous: Nothing else other than y, derivatives of y, and their coefficients
- This is the easiest type of differential equation

3.1 Direction Fields

- Derivatives bear the physical meaning of "(slope)"
- In the xy-plane, the slope is talking about the z-axis (which can be anything like E-field strength or height of a hill)
- We can draw direction fields to represent the (slope) of a differential equation
- Example:
  In a field, there are x rabbits and some wolves. In a month and in absence of predators, population of a rabbit is grow in proportional to its population, i.e. $\frac{dx}{dt} = rx$ for some contant r. Assume that population of wolves do not change and they kill s rabbits per month. Find the equation to describe the population of rabbits.
  - Modeling: Change of population in a month is

$$\frac{dx}{dt} = rx - s$$

  - Solving:

$$\begin{aligned} \frac{dx}{rx - s} &= dt \\ \frac{1}{r}\ln(rx - s) &= t + C' \\ rx - s &= e^{rt+C} \\ x &= \frac{1}{r}\left(e^{rt+C} + s\right) \end{aligned}$$

  - This is a way to solve it, but we cannot see what the above equation represents physically

- If we represents the equation in a picture, it can be easier to get a sense of the solution: (read the supplementary material)
- Way to draw the direction field: (Hay, watch me!)

  1. Draw isoclines, by replacing $\frac{dx}{dt}$ with different numbers, e.g. 0, 1, 2, -1, -2, ...
  2. On the isoclines of $\frac{dx}{dt} = k$, draw many little arrows corresponding to the slope k
  3. Draw some smooth lines that joins different arrows up. These are approximate solutions

- There can be many different approximate solutions on the direction field, because we may have different (constants) of integation

  - Selecting the appropriate constant: (initial-value) problem

3.2 Separable Equations

- An differential equation separable if we can put x and y onto (different) sides, e.g.:

$$\begin{aligned} g(y)dy &= f(x)dx \\ \int g(y)dy &= \int f(x)dx \\ G(y) &= F(x) + C \\ y &= G^{-1}(F(x) + C) \end{aligned}$$

- This type is easy to solve, for example, Problem Set 1.3 Question 12:
  Solve the initial value problem: $y' = -x/y, \quad y(1) = \sqrt{3}$

$$\begin{aligned} \frac{dy}{dx} &= -\frac{x}{y} \\ (ydy &= -xdx) \qquad \leftarrow \text{separable form} \\ \frac{1}{2}y^2 &= -\frac{1}{2}x^2 + C' \\ y &= \sqrt{C - x^2} \end{aligned}$$

  For the initial value part, substitute $x = 1$, we get $y = \sqrt{C-1}$, thus $\sqrt{C-1} = \sqrt{3}$ or $C = 4$. Thus

$$y = \sqrt{4 - x^2}$$

- But sometimes, we may not get it separable directly. Hence we need some way to convert the equation into separable form.

  - General rule: by (substitution)
  - In our assignment, we used the substitution $u = y/x$ and $v = ay + bx + k$

- Example of $u = y/x$: Problem Set 1.3 Question 20

Setting $y/x = u$, solve the initial value problem: $xy' = (y-x)^3 + y, \quad y(1) = 3/2$

$$\begin{aligned}
xy' &= (y-x)^3 + y \\
(x(xu)' &= (xu-x)^3 + xu) \\
x(x'u + u'x) &= (xu-x)^3 + xu \\
xu + x^2u' &= x^3(u-1)^3 + xu \\
\frac{u'}{(u-1)^3} &= x \\
\int (u-1)^{-3} du &= \int x dx \\
-\frac{1}{2}(u-1)^{-2} &= \frac{1}{2}x^2 + C \\
\frac{x^2}{y^2 - 2xy + x^2} &= -x^2 + C \qquad \leftarrow \text{remember to subst. back}
\end{aligned}$$

Substituting $x = 1$ and $y = \frac{3}{2}$, we have:

$$\begin{aligned}
\frac{1}{\frac{9}{4} - 3 + 1} &= -1 + C \\
4 + 1 &= C \\
C &= 5
\end{aligned}$$

Substitute C back to the above equation and simplify, you will get the answer (implicit form is better-looking)

- Example of $v = ay + bx + k$: Problem Set 1.3 Question 24
  Solve $y' = (x+y-2)^2$.

$$\begin{aligned}
v &= x + y - 2 \\
\therefore\ v' &= (1 + y') \\
y' &= (x+y-2)^2 \\
\Longrightarrow \quad v' - 1 &= v^2 \\
\frac{v'}{v^2 + 1} &= 1 \\
\arctan v &= x + C \\
\arctan(x+y-2) &= x + C \\
\arctan(x+y-2) - x + C' &= 0
\end{aligned}$$

3.3 How to Create Your Own Equation?

- Differential equation is very useful, especially if you want to do optimization or model a system
- We model a system using differential equation is easier then model it explicitly because we are just describing its (behavior)
- Example: Problem Set 1.4 Question 14
  (Mixing problem) A tank contains 400 gal of brine in which 100 lb of salt are dissolved. Fresh water runs into the tank at the rate of 2 gal/min, and the mixture, kept practically uniform by stirring, runs out at the same rate. How much salt will there be in the tank at the end of 1 hour?
  - What we want: The equation describing how much salt in the tank
  - How to do:

1. "runs out at the same rate" $\rightarrow$ water in = (water out)
2. Salt in = 0
3. Salt out = amount of salt in 2 gal
   If we set the amount of salt in the tank represented by y, the salt desity is $y/400$ lb per gal
4. Hence the equation is:
   $$\frac{dy}{dx} = -2\left(\frac{y}{400}\right) = \frac{-y}{200}$$
   with the initial condition $y(0) = 100$.

- More complex system may comes out a more complex differential equation (e.g. non-linear, higher order, non-algebraic)

4 Exact Differential Equation

- Another class of differential equations that we can solve is the exact differential equation.
- It is due to the fact that, if we have a function $u(x, y)$, its differential is
  $$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$
  If we found a differential equation of the above form with $du = 0$, we call it exact differential equation
- How to know if it is exact or not?
  - Criteria 1: Differential equation looks like: $M(x, y)dx + N(x, y)dy = 0$
  - Criteria 2: M and N are really complementary partial derivatives, i.e.
    $$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}$$
- Example: Problem Set 1.5, Question 20, $(2xydx + dy)e^{x^2} = 0, \qquad y(0) = 2$
  - Check for exact:
    $$\begin{aligned} 2xye^{x^2}dx + e^{x^2}dy &= 0 \\ \partial_y\left(2xye^{x^2}\right) &= (2xe^{x^2}) \\ \partial_x\left(e^{x^2}\right) &= (2xe^{x^2}) \end{aligned}$$
  - Solve it:
    $$\begin{aligned} \frac{\partial u}{\partial y} &= e^{x^2} \\ u(x, y) &= ye^{x^2} + h(x) \\ \frac{\partial u}{\partial x} &= 2xye^{x^2} + h'(x) \\ 2xye^{x^2} &= 2xye^{x^2} + h'(x) \\ h'(x) &= (0) \\ u(x, y) &= ye^{x^2} + C' \\ 0 &= ye^{x^2} + C \end{aligned}$$

Substitute $x = 0$, $y = 2$:

$$\begin{aligned} 0 &= 2 + C \\ \therefore \quad ye^{x^2} - 2 &= 0 \end{aligned}$$

- If a differential equation is not exact, we may try to make it exact
  - By multiplying a function called the (integrating factors)
  - In hope that, with function $F(x, y)$, we have:

$$\begin{aligned} F(x,y)M(x,y)dx + F(x,y)N(x,y)dy &= 0 \\ \partial_y\left(F(x,y)M(x,y)\right) &= \partial_x\left(F(x,y)N(x,y)\right) \end{aligned}$$

  - For simplicity, we usually assume $F(x)$ or $F(y)$ only, i.e. single-variable factors, and they are:

$$\begin{aligned} F(x) &= \exp\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx \\ F(y) &= \exp\int \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dy \end{aligned}$$

- Example: Problem Set 1.5 Question 32, solve $2xydx + 3x^2dy = 0$.

$$\begin{aligned} F(x) &= \exp\int \frac{1}{3x^2}\left(2x - 6x\right)dx \\ &= \exp\int \left(\frac{-4}{3x}\right)dx \\ &= \exp\left(-\frac{4}{3}\ln x\right) \\ &= x^{-4/3} \end{aligned}$$

  - Hence, the equation becomes:

$$\begin{aligned} x^{-4/3}2xydx + x^{-4/3}3x^2dy &= 0 \\ 2x^{-1/3}ydx + 3x^{2/3}dy &= 0 \\ \int 3x^{2/3}dy &= 3x^{2/3}y + h(x) = u(x,y) \\ \partial_x u(x,y) &= (2x^{-1/3}y + h'(x)) \\ \therefore \quad h'(x) &= 0 \\ u(x,y) &= 3x^{2/3}y + C' \\ 3x^{2/3}y + C &= 0 \end{aligned}$$

5 Linear Differential Equations

5.1 General Solutions

- Yet another class of differential equation that we can solve is called the Linear Differential Equation:

$$y' + p(x)y = r(x)$$

If $r(x) \equiv 0$, the equation is called homogeneous, and the solution is given by:

$$y(x) = Ce^{-h}, \qquad \text{where } h = \int p(x)dx$$

But if it is non-homogeneous, the solution is a bit more complicated:

$$y(x) = e^{-h}\left[\int e^h r dx + C\right], \qquad \text{where } h = \int p(x)dx$$

– It is very easy to derive the above formulae, see book page 33-34 (by using integrating factors)

- Example: Problem Set 1.6 Question 18
  Solve the following initial value problem: $y' = y \tan x, \quad y(\pi) = 2$

$$\begin{aligned} y' - (\tan x)y &= 0 \\ y(x) &= Ce^{-h} \\ \text{where } h &= \int -\tan x dx \\ &= \ln|\cos x| \\ y(\pi) &= Ce^{-\ln|\cos \pi|} \\ &= Ce^0 \\ C &= 2 \\ y(x) &= 2e^{-\ln|\cos x|} \end{aligned}$$

- Example: Problem Set 1.6 Question 8, $y' + 4y = \cos x$

$$\begin{aligned} y' + 4y &= \cos x \\ y(x) &= e^{-h}\left[\int e^h r dx + C\right] \\ \text{where } h &= (\int 4dx = 4x) \\ y(x) &= e^{-4x}\left[\int e^{4x} \cos x dx + C\right] \\ \text{But } \int e^{4x} \cos x dx &= \int e^{4x} d(\sin x) \\ &= e^{4x} \sin x - 4\int \sin x e^{4x} dx \\ &= e^{4x} \sin x + 4\int e^{4x} d(\cos x) \\ &= e^{4x} \sin x + 4(\cos x e^{4x} - 4\int \cos x e^{4x} dx) \\ &= e^{4x} \sin x + 4e^{4x} \cos x - 16\int e^{4x} \cos x dx \\ 17\int e^{4x} \cos x dx &= e^{4x}(\sin x + 4\cos x) \\ \therefore \quad y(x) &= e^{-4x}\left[\frac{1}{17}e^{4x}(\sin x + 4\cos x) + C\right] \\ &= \frac{1}{17}(\sin x + 4\cos x) + Ce^{-4x} \end{aligned}$$

- The homogeneous/non-homogeneous linear differential equations have many properties

  – See assignment Problem Set 1.6, Questions 23-30.

5.2 Bernoulli Equation

- Several non-linear differential equations can be converted into linear form. One of them is called the Bernoulli equations:

$$y' + p(x)y = g(x)y^a$$

- Solution: By substitution of $u = y^{1-a}$, we can convert the above equation into

$$u' + (1-a)p(x)u = (1-a)g(x)$$

  which can be solved using the method in the previous subsection.

- Example: Problem Set 1.6 Question 32, $y' + y = -x/y$.

$$\begin{aligned}
y' + y &= -x/y \\
yy' + y^2 &= -x \\
\text{We define } u &= y^2 \\
u' &= 2yy' \\
\text{then, } u' + 2u &= -2x \\
u(x) &= e^{-h}\left[\int e^h(-2x)dx + C\right] \\
\text{where } h &= \int dx = x \\
\therefore\ u(x) &= e^{-x}\left[-2\int xe^x dx + C\right] \\
&= e^{-x}\left[-2(xe^x - e^x) + C\right] \\
&= -2(x-1) + Ce^{-x} \\
y &= \pm\sqrt{Ce^{-x} - 2(x-x)}
\end{aligned}$$