ERG2011A Tutorial 7: Second-Order Linear Differential Equations

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1 Previous knowledge

- Separable equations: (sometimes need substitution like $u = y/x$ or $v = ay + bx + k$)

$$\begin{aligned} g(y)dy &= f(x)dx \\ \implies \quad y &= G^{-1}(F(x) + C) \end{aligned}$$

- Exact differential equation:

$$M(x,y)dx + N(x,y)dy = 0 \quad \text{with} \quad \frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

which solve by

$$\begin{aligned} \int M(x,y)dx &= f(x,y) + h(y) \\ \frac{\partial}{\partial y} f(x,y) + h'(y) &= N(x,y) \end{aligned}$$

- Use of integrating factors:

$$\begin{aligned} F(x,y)M(x,y)dx + F(x,y)N(x,y)dy &= 0 \\ \partial_y \left(F(x,y)M(x,y)\right) &= \partial_x \left(F(x,y)N(x,y)\right) \end{aligned}$$

if F is single variable, it is:

$$\begin{aligned} F(x) &= \exp \int \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) dx \\ F(y) &= \exp \int \frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dy \end{aligned}$$

- Linear differential equation:

$$\begin{aligned} y' + p(x)y &= r(x) \\ \implies \quad y(x) &= e^{-h} \left[\int e^h r dx + C \right], \text{ where } h = \int p(x)dx \end{aligned}$$

If $r(x) \equiv 0$, $y(x) = Ce^{-h}$

- Bernoulli equation: (solve by substitution of $u = y^{1-a}$)

$$\begin{aligned} y' + p(x)y &= g(x)y^a \\ \implies \quad u' + (1-a)p(x)u &= (1-a)g(x) \end{aligned}$$

2 Second-order Linear Equation

- Standard form of L.D.E.:

$$y'' + p(x)y' + q(x)y = r(x)$$

  If $r(x) \equiv 0$, we call it homogeneous, otherwise it is non-homogeneous. $p(x)$, $q(x)$, $r(x)$ can be anything independent of y.

- Be careful that usually, differential equation gives many solutions if we do not constrain it with initial values or boundary values

- Properties:

  1. If y_1 and y_2 are solutions of $y'' + p(x)y' + q(x)y = 0$, then $c_1y_1 + c_2y_2$ is also a solution $\forall c_1, c_2 \in \mathbb{R}$
  2. If y_1/y_2 is not a real number, they are called independent solutions.
     Then the general solutions of $y'' + p(x)y' + q(x)y = 0$ is $c_1y_1 + c_2y_2$ $\forall c_1, c_2 \in \mathbb{R}$.
  3. If y_3 is a particular solution of $y'' + p(x)y' + q(x)y = r(x)$, then the general solution of it is $c_1y_1 + c_2y_2 + y_3$ $\forall c_1, c_2 \in \mathbb{R}$. In other words, the general solution of homogeneous equation plus any solution of non-homogeneous equation is the general solution of nonhomogeneous equation.

- A better way of determining independence of solutions is using the Wronski determinant:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$$

  $W(y_1, y_2) \neq 0$ for some values of x if and only if y_1 and y_2 are linearly independent

  - Reference: Book section 2.7

- Solving a homogeneous equation is therefore a critical part for general solution

3 Homogeneous Linear Equation

3.1 General coefficients

- Solving $y'' + p(x)y' + q(x)y = 0$ generally can be difficult sometimes

  - Hence we may find the first solution y_1 by guessing and trials

- After y_1 is found, we can find y_2 using the method of reduction of order

  1. Let $y_2 = u(x, y)y_1$
  2. Finally we find that: $u(x, y) = \displaystyle\int U dx$, where $U = \dfrac{1}{y_1^2} \exp\left(-\displaystyle\int p(x)dx\right)$
     - Reference: Book page 69-70

- If we found an homogeneous non-linear equation, we can convert it into linear equation sometimes:

  - $F(x, y', y'') = 0 \quad \Longrightarrow \quad$ substitute $z = y'$
  - $F(y, y', y'') = 0 \quad \Longrightarrow \quad$ substitute $z = y'$, $y'' = z\dfrac{dz}{dy}$

- Example: Problem Set 2.1 Question 8: $xy'' + y' = 0$

$$\begin{aligned}
xy'' + y' &= 0 \\
x &= -y'/y'' \\
\frac{1}{x} &= -\frac{y''}{y'} \\
\int \frac{dx}{x} &= -\int \frac{d(y')}{y'} \\
\ln x &= -\ln(y') \\
x &= \frac{1}{y'} \\
y' &= \frac{1}{x} \\
y &= \int \frac{dx}{x} \\
&= \ln x
\end{aligned}$$

Now we found $y_1 = \ln x$. Because the question is equivalent to $y'' + y'/x = 0$, we can have

$$\begin{aligned}
U &= \frac{1}{y_1^2} \exp\left(-\int \frac{1}{x} dx\right) \\
&= \frac{1}{(\ln x)^2} \exp(-\ln x) \\
&= \frac{1}{x(\ln x)^2} \\
u &= \int \frac{1}{x(\ln x)^2} dx \\
&= \int \frac{1}{(\ln x)^2} d(\ln x) \\
&= -\frac{1}{\ln x} \\
y_2 &= uy_1 = -1
\end{aligned}$$

Hence the general solution for $xy'' + y' = 0$ is $y = c_1 \ln x - c_2$ or $y = c_1 \ln x + c_3$. Alternatively, by taking care of the constants of integration, we can solve it in one-pass (just because we are lucky to have such simple equation):

$$\begin{aligned}
xy'' + y' &= 0 \\
x &= -y'/y'' \\
\frac{1}{x} &= -\frac{y''}{y'} \\
\int \frac{dx}{x} &= -\int \frac{d(y')}{y'} \\
\ln x + C_1 &= -\ln(y') \\
e^{C_1} x &= \frac{1}{y'} \\
y' &= e^{-C_1} \frac{1}{x} \\
y &= e^{-C_1} \int \frac{dx}{x} \\
&= e^{-C_1} \ln x + C_2 e^{-C_1} \\
&= c_1 \ln x + c_2
\end{aligned}$$

- Example: Problem Set 2.1 Question 10: $y'' + (1 + y^{-1})y'^2 = 0$

$$\begin{aligned}
y'' + (1 + \frac{1}{y})y'^2 &= 0 \\
z\frac{dz}{dy} + (1 + \frac{1}{y})z^2 &= 0 \\
yz\frac{dz}{dy} + (y+1)z^2 &= 0 \\
y\frac{dz}{dy} + (y+1)z &= 0 \\
\frac{y+1}{y}dy &= -\frac{dz}{z} \\
y + \ln y + C_0 = \int (1 + \frac{1}{y})dy &= -\int \frac{dz}{z} \quad = -\ln z \\
C_1 y e^y &= \frac{1}{z} = \frac{dx}{dy} \\
C_1 \int (ye^y)dy &= x \\
C_1(y-1)e^y + C_2 &= x
\end{aligned}$$

3.2 Constant coefficients

- If the equation is: $y'' + ay' + by = 0$, then:
  1. $y = e^{\lambda x}$ is a solution
  2. Where λ satisfies the quadratic equation $\lambda^2 + a\lambda + b = 0$
     we call this quadratic equation the characteristic equation
- If $\Delta = a^2 - 4b$ is:
  1. $\Delta > 0$, then $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \quad \forall c_1, c_2 \in \mathbb{R}$
  2. $\Delta = 0$, then $y = (c_1 + c_2 x)e^{-ax/2} \quad \forall c_1, c_2 \in \mathbb{R}$, whereas $\lambda = -a/2$
  3. $\Delta < 0$, then $y = e^{-ax/2}(c_1 \cos \omega x + c_2 \sin \omega x) \quad \forall c_1, c_2 \in \mathbb{R}$, whereas $\omega = \sqrt{b - \frac{1}{4}a^2}$ and $\lambda = -\frac{1}{2}a \pm i\omega$
- Example: Problem Set 2.2 Question 8: $10y'' + 6y' - 4y = 0$

$$\begin{aligned}
10y'' + 6y' - 4y &= 0 \\
y'' + \frac{3}{5}y' - \frac{2}{5}y &= 0 \\
\therefore \quad \Delta &= \frac{9}{25} - 4\left(\frac{-2}{5}\right) = \frac{17}{25} > 0 \\
\text{Solving} \quad 10\lambda^2 + 6\lambda - 4 &= 0 \\
\text{Gives} \quad \lambda &= -1 \text{ or } \frac{2}{5} \\
\therefore \quad y &= c_1 e^{-x} + c_2 e^{2x/5} \quad \forall c_1, c_2 \in \mathbb{R}
\end{aligned}$$

- Example: Problem Set 2.2 Question 9: $y'' + 2ky' + k^2 y = 0$

$$\begin{aligned}
y'' + 2ky + k^2 y &= 0 \\
\implies \quad \Delta &= (2k)^2 - 4(k^2) = 0 \\
\therefore \quad y &= (c_1 + c_2 x)e^{-2kx/2}
\end{aligned}$$

- Example: Problem Set 2.3 Question 10: $y'' - 2\sqrt{2}y' + 2.5y = 0$

$$\begin{aligned}
y'' - 2\sqrt{2}y' + 2.5y &= 0 \\
\Longrightarrow \quad \Delta &= 8 - 4(2.5) < 0 \\
\therefore \quad \omega &= \sqrt{2.5 - \frac{1}{4}(-2\sqrt{2})^2} \\
&= \sqrt{2.5 - \frac{8}{4}} = \sqrt{0.5} \\
&= \frac{1}{4} \\
\therefore \quad y &= e^{\sqrt{2}x}\left(c_1 \cos\frac{x}{4} + c_2 \sin\frac{x}{4}\right)
\end{aligned}$$

- Knowing these technique can help you solve any cases in damping motions (see book section 2.5)

3.3 Partial constant coefficients — Euler-Cauchy Equations

- Euler-Cauchy equations have some of the coefficients are constant:

$$x^2y'' + axy' + by = 0$$

- A particular solution is given by:

  1. $y = x^m$
  2. Where m satisfies the quadratic equation: $m^2 + (a-1)m + b = 0$

- With different cases of the discriminants $\Delta = (a-1)^2 - 4b$,

  1. $\Delta > 0$, then $y = c_1x^{m_1} + c_2x^{m_2}$
  2. $\Delta = 0$, then $y = (c_1 + c_2 \ln x)x^{(1-a)/2}$
  3. $\Delta < 0$, then $y = x^{\mu}\left[c_1 \cos(\nu \ln x) + c_2 \sin(\nu \ln x)\right]$, where $m = \mu \pm i\nu = (1-a) \pm i\left(4b - (a-1)^2\right)$.

- Example: Problem Set 2.6 Question 8: $(xD^2 + D)y = 0$

$$\begin{aligned}
(xD^2 + D)y &= 0 \\
xD^2y + Dy &= 0 \\
xy'' + y' &= 0 \\
x^2y'' + xy' + 0 \cdot y &= 0 \quad \Longrightarrow \quad a = 1, b = 0 \\
\therefore \quad \Delta &= (1-1)^2 - 4(0) = 0 \\
\therefore \quad y &= (c_1 + c_2 \ln x)x^{(1-1)/2} \\
&= c_1 + c_2 \ln x
\end{aligned}$$

- Example: Problem Set 2.6 Question 10: $(x^2D^2 + 0.7xD - 0.1)y = 0$

$$\begin{aligned}
(x^2D^2 + 0.7xD - 0.1)y &= 0 \\
x^2y'' + 0.7xy' - 0.1y &= 0 \\
\therefore \quad \Delta &= (0.7-1)^2 - 4(-0.1) > 0 \\
\text{and} \quad m &= 0.5 \text{ or } -0.2 \\
\therefore \quad y &= c_1x^{0.5} + c_2x^{-0.2}
\end{aligned}$$

4 Non-Homogeneous Linear Equations

4.1 Constant coefficients

- Solving non-homogeneous equations can be difficult. So we look at a simplier case with constant coefficients:

$$y'' + ay' + by = r(x)$$

- Rule of thumb: $y = y_h + y_p$
(General solution is the composition of homogeneous general solution and a particular solution)

- How to find y_p? We use the method of undetermined coefficients:

$r(x)$	y_p
$ke^{\gamma x}$	$Ce^{\gamma x}$
kx^n $(n = 0, 1, \ldots)$	Polynomial: $\sum\limits_{i=0}^{n} K_i x^i$
$k\cos\omega x$ or $k\sin\omega x$	$K\cos\omega x + M\sin\omega x$
$ke^{\alpha x}\cos\omega x$ or $ke^{\alpha x}\sin\omega x$	$e^{\alpha x}(K\cos\omega x + M\sin\omega x)$

- Rules:

1. If $r(x)$ is as shown in the above table, choose the corresponding y_p
2. If y_p chosen is already represented by y_h, modify y_p by multiplying x; or multiplying x^2 if y_h is obtained with double root of λ
3. If $r(x)$ is a sum of several functions above, choose $y_p$to be the sum of corresponding functions accordingly

- Example: Problem Set 2.8 Question 8
Verify y_p is a particular solution and find the general solution: $(8D^2 - 6D + 1)y = 6\cosh x, \quad y_p = \frac{1}{5}e^{-x} + e^x$

  - Verification:

$$\begin{aligned}
y_p &= \frac{1}{5}e^{-x} + e^x \\
y_p' &= -\frac{1}{5}e^{-x} + e^x \\
y_p'' &= \frac{1}{5}e^{-x} + e^x \\
\therefore \quad 8y_p'' - 6y_p' + y_p &= 8(\frac{1}{5}e^{-x} + e^x) - 6(-\frac{1}{5}e^{-x} + e^x) + \frac{1}{5}e^{-x} + e^x \\
&= \frac{8+6+1}{5}e^{-x} + (8 - 6 + 1)e^x \\
&= 3e^{-x} + 3e^x \\
&= 6 \cdot \frac{e^{-x} + e^x}{2} = 6\cosh x
\end{aligned}$$

  - General solution:

$$\begin{aligned}
8y'' - 6y' + y &= 0 \\
\Longrightarrow \quad m &= \frac{1}{2} \text{ or } \frac{1}{4} \\
\therefore \quad y_h &= c_1 e^{x/2} + c_2 e^{x/4} \\
\therefore \quad y &= c_1 e^{x/2} + c_2 e^{x/4} + \frac{1}{5}e^{-x} + e^x \quad \forall c_1, c_2 \in \mathbb{R}
\end{aligned}$$

- Example: Problem Set 2.9 Question 8: $y'' + 6y' + 9y = 50e^{-x}\cos x$
  - Homogeneous solution, y_h

$$\begin{aligned} y'' + 6y' + 9y &= 0 \\ \implies \quad \Delta &= 6^2 - 4 \cdot 9 = 0 \\ \therefore \quad y_h &= (c_1 + c_2 x)e^{-3x} \quad \forall c_1, c_2 \in \mathbb{R} \end{aligned}$$

  - Particular solution, y_p

$$\begin{aligned} r(x) &= 50e^{-x}\cos x \\ \implies \quad y_p &= e^{-x}(K\cos x + M\sin x) \\ \therefore \quad y_p' &= -e^{-x}(K\cos x + M\sin x) + e^{-x}(-K\sin x + M\cos x) \\ y_p'' &= e^{-x}(K\cos x + M\sin x) - e^{-x}(-K\sin x + M\cos x) \\ &\quad -e^{-x}(-K\sin x + M\cos x) + e^{-x}(-K\cos x - M\sin x) \\ &= 2e^{-x}(K\sin x - M\cos x) \\ \therefore \quad y_p'' + 6y_p' + 9y_p &= 2e^{-x}(K\sin x - M\cos x) \\ &\quad -6e^{-x}(K\cos x + M\sin x) + 6e^{-x}(-K\sin x + M\cos x) \\ &\quad +9e^{-x}(K\cos x + M\sin x) \\ &= 3e^{-x}(K\cos x + M\sin x) - 4e^{-x}(K\sin x - M\cos x) \\ &= (3K + 4M)e^{-x}\cos x + (3M - 4K)e^{-x}\sin x \\ \text{hence,} \quad r(x) = 50e^{-x}\cos x \quad &\implies \begin{cases} 3K + 4M = 50 \\ 3M - 4K = 0 \end{cases} \\ &\implies \begin{cases} K = \frac{25}{3} \\ M = \frac{25}{4} \end{cases} \\ \therefore \quad y_p &= e^{-x}\left(\frac{25}{3}\cos x + \frac{25}{4}\sin x\right) \end{aligned}$$

  - General soluton: $y_h + y_p$

$$y = (c_1 + c_2 x)e^{-3x} + e^{-x}\left(\frac{25}{3}\cos x + \frac{25}{4}\sin x\right) \quad \forall c_1, c_2 \in \mathbb{R}$$

4.2 Method of variation of parameters

- If the coefficient is not constant, we solve it by the method of variation of parameters
  - Caution: Complicated but almighty
- Definitions:
  - Given a homogeneous counterpart of the equation, setting $c_1 = 1,\ c_2 = 0$ and $c_1 = 0,\ c_2 = 1$ resepctively will give out y_1 and y_2. We call them the basis
  - Wronskian: $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$
  - Particular solution:

$$y_p = -y_1 \int \frac{y_2 r(x)}{W} dx + y_2 \int \frac{y_1 r(x)}{W} dx$$

- Example: Problem Set 2.10 Question 8: $(D^2 + 6D + 9)y = 16e^{-3x}/(x^2 + 1)$
  - $r(x)$ does not look like sum of things we know — cannot find y_p by looking up the table
  - Solve homogeneous counterpart: as in Problem Set 2.9 Question 8

$$\begin{aligned} y_h &= (c_1 + c_2 x)e^{-3x} \quad \forall c_1, c_2 \in \mathbb{R} \\ \therefore \quad y_1 &= e^{-3x} \\ y_2 &= xe^{-3x} \end{aligned}$$

  - Wronskian:

$$\begin{aligned} W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1' \\ &= e^{-3x} \cdot \left(e^{-3x} - 3xe^{-3x}\right) - xe^{-3x} \cdot \left(-3e^{-3x}\right) \\ &= e^{-6x} - 3xe^{-6x} + 3xe^{-6x} \\ &= e^{-6x} \end{aligned}$$

  - Particular solution:

$$\begin{aligned} y_p &= -y_1 \int \frac{y_2 r(x)}{W} dx + y_2 \int \frac{y_1 r(x)}{W} dx \\ &= -e^{-3x} \int \frac{xe^{-3x} \cdot 16e^{-3x}/(x^2+1)}{e^{-6x}} dx + xe^{-3x} \int \frac{e^{-3x} \cdot 16e^{-3x}/(x^2+1)}{e^{-6x}} dx \\ &= -e^{-3x} \int \frac{16x}{x^2+1} dx + xe^{-3x} \int \frac{16}{x^2+1} dx \\ &= -16e^{-3x} \cdot \frac{1}{2} \ln(x^2+1) + 16xe^{-3x} \cdot \tan^{-1}(x) \\ &= e^{-3x} \left(16x \tan^{-1} x - 8 \ln(x^2+1)\right) \end{aligned}$$

  - General solution:

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 + c_2 x)e^{-3x} + e^{-3x} \left(16x \tan^{-1} x - 8 \ln(x^2+1)\right) \\ &= e^{-3x} \left(c_1 + c_2 x + 16x \tan^{-1} x - 8 \ln(x^2+1)\right) \end{aligned}$$

- Example: Problem Set 2.10 Question 14: $(x^2 D^2 - 4xD + 6)y = 7x^4 \sin x$
  - Solving homogeneous version: (this is an Eular-Cauchy Equation)

$$\begin{aligned} x^2 y'' - 4xy' + 6y &= 0 \\ \implies \quad \Delta &= (-4-1)^2 - 4(6) = 25 - 24 > 0 \\ \therefore \quad m &= 2 \text{ or } 3 \\ \therefore \quad y_h &= c_1 x^2 + c_2 x^3 \end{aligned}$$

  - Wronskian:

$$\begin{aligned} W &= y_1 y_2' - y_2 y_1' \\ &= x^2 \cdot \left(3x^2\right) - x \cdot (2x) \\ &= 3x^4 - 2x^4 \\ &= x^4 \end{aligned}$$

– Particular solution:

$$\begin{aligned}
y_p &= -y_1\int \frac{y_2 r(x)}{W}dx + y_2\int \frac{y_1 r(x)}{W}dx \\
&= -x^2\int \frac{x^3\cdot 7x^4\sin x}{x^4}dx + x^3\int \frac{x^2\cdot 7x^4\sin x}{x^4}dx \\
&= -7x^2\int x^3\sin x dx + 7x^3\int x^2\sin x dx \quad \leftarrow \text{solve this using "integration by part"} \\
&= -7x^2\left((6x-x^3)\cos x + (3x^2-6)\sin x\right) + 7x^3\left((2-x^2)\cos x + 2x\sin x\right) \\
&= -7x^2\left((-x^3-x^2+6x+2)\cos x + (3x^2+2x-6)\sin x\right)
\end{aligned}$$

– General solution:

$$\begin{aligned}
y &= y_h + y_p \\
&= c_1x^2 + c_2x^3 - 7x^2\left((-x^3-x^2+6x+2)\cos x + (3x^2+2x-6)\sin x\right)
\end{aligned}$$

– How to use integration by part to solve $\int x^2\sin x dx$ and $\int x^3\sin x dx$? — in case you forgot about that

$$\begin{aligned}
\int x^2\sin x dx &= -\int x^2 d(\cos x) \\
&= -x^2\cos x - (-\int \cos x d(x^2)) \\
&= -x^2\cos x + 2\int x\cos x dx \\
&= -x^2\cos x + 2\int x d(\sin x) \\
&= -x^2\cos x + 2x\sin x - 2\int \sin x dx \\
&= -x^2\cos x + 2x\sin x + 2\cos x
\end{aligned}$$

$$\begin{aligned}
\int x^3\sin x dx &= -\int x^3 d(\cos x) \\
&= -x^3\cos x - (-\int \cos x d(x^3)) \\
&= -x^3\cos x + 3\int x^2\cos x dx \\
&= -x^3\cos x + 3\int x^2 d(\sin x) \\
&= -x^3\cos x + 3x^2\sin x - 3\int \sin x d(x^2) \\
&= -x^3\cos x + 3x^2\sin x - 6\int x\sin x dx \\
&= -x^3\cos x + 3x^2\sin x + 6\int x d(\cos x) \\
&= -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\int \cos x dx \\
&= -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x
\end{aligned}$$

5 Higher-order Linear Equations with Constant Coefficients

- Standard form:

$$\frac{d^n y}{dx^n} + p_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_1(x)\frac{dy}{dx} + p_0(x)y = r(x)$$

- The Wronskian in Second-order linear differential equation is a 2×2 matrix, but here, it is a $n \times n$ matrix:

$$W(y_1, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

- We don't deal with the cases that coefficients of $\dfrac{d^n y}{dx^n}$ are not a real number in this course, i.e., we only handle

$$y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_1 y' + p_0 y = r(x)$$

5.1 Homogeneous Equations

- Similar to second-order cases, we have the characteristic equation: $\lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_1\lambda + p_0 = 0$
  - Roots of the characteristic equation are $\lambda_1, \lambda_2, \ldots, \lambda_n$
  - Order-n equations have n basis in any cases
- If all n roots are distinct, we have n basis: $y_k = e^{\lambda_k x},\ k = 1, \ldots, n$
- If some of the roots are repeated m times, we have m of the n basis in the form: $y_k = x^k e^{\lambda x},\ k = 0, \ldots, m-1$
- If some the roots are a pair of complex conjugates, $\lambda = \gamma \pm i\omega$, we have 2 of the n basis in the form: $y_1 = e^{\gamma x}\cos\omega x$, $y_2 = e^{\gamma x}\sin\omega x$
- If some the roots are repeated complex conjugates repeated m times, we have $2m$ of the n basis in the form $y_k = x^k e^{\gamma x}\cos\omega x,\ y_{m+k} = x^k e^{\gamma x}\sin\omega x,\ k = 0, \ldots, m-1$

5.2 Non-Homogeneous Equations

- Follow the method as in second-order, except the general solution for homogeneous counterpart, y_h, is consisting of n components and n arbitrary constants
- $y = y_h + y_p = c_1y_1 + c_2y_2 + \cdots + c_ny_n + y_p$
- Easy, no magic!

6 Summary of Second-order Linear D.E.

- Standard form of L.D.E.: $y'' + p(x)y' + q(x)y = r(x)$
- Properties of L.D.E.:
  1. If y_1 and y_2 are solutions of the homo LDE and $y_1/y_2 \neq$constant, they are independent solutions
  2. If y_1 and y_2 are independent solutions of the homo LDE, then $c_1y_1 + c_2y_2$ is the general solution
  3. If y_3 is a particular solution of non-homo LDE, then the general solution of it is $c_1y_1 + c_2y_2 + y_3$
- Wronski determinant:$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$
  - $W(y_1, y_2) \not\equiv 0 \iff y_1$ and y_2 are linearly independent
- If y_1 is one solution of a homo LDE, then $y_2 = y_1 \int \frac{1}{y_1^2} e^{-\int p(x)dx} dx$ (method of reduction of order)
- Substitution for conversion to homo LDE:
  - $F(x, y', y'') = 0 \implies$ substitute $z = y'$
  - $F(y, y', y'') = 0 \implies$ substitute $z = y'$, $y'' = z\frac{dz}{dy}$
- Special homo LDE:

	Constant coefficient	Eular-Cauchy equation
	$y'' + ay' + by = 0$	$x^2y'' + axy' + by = 0$
Char. eqn	$\lambda^2 + a\lambda + b = 0$	$m^2 + (a-1)m + b = 0$
Δ	$a^2 - 4b$	$(a-1)^2 - 4b$
$\Delta > 0$	$y = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x}$	$y = c_1x^{m_1} + c_2x^{m_2}$
$\Delta = 0$	$y = (c_1 + c_2x)e^{-ax/2}$	$y = (c_1 + c_2 \ln x)x^{(1-a)/2}$
$\Delta < 0$	$\lambda = -\frac{1}{2}a \pm i\omega = -\frac{1}{2}a \pm i\sqrt{b - \frac{1}{4}a^2}$ $y = e^{-ax/2}(c_1 \cos\omega x + c_2 \sin\omega x)$	$m = \mu \pm i\nu = (1-a) \pm i\left(4b - (a-1)^2\right)$ $y = x^\mu \left[c_1 \cos(\nu \ln x) + c_2 \sin(\nu \ln x)\right]$

- Non-homo L.D.E.: Guess y_p and then solve for the unknown coefficients:
  1. If $r(x)$ is as shown in the table, choose the corresponding y_p
  2. If y_p chosen is already represented by y_h, modify y_p by multiplying x; or multiplying x^2 if y_h is obtained with double root of λ
  3. If $r(x)$ is a sum of several functions above, choose $y_p$to be the sum of corresponding functions accordingly

$r(x)$	y_p
$ke^{\gamma x}$	$Ce^{\gamma x}$
kx^n $(n = 0, 1, \ldots)$	Polynomial: $\sum_{i=0}^{n} K_ix^i$
$k\cos\omega x$ or $k\sin\omega x$	$K\cos\omega x + M\sin\omega x$
$ke^{\alpha x}\cos\omega x$ or $ke^{\alpha x}\sin\omega x$	$e^{\alpha x}(K\cos\omega x + M\sin\omega x)$

- Method of variation of parameters: for use when $r(x)$ is not in the above table
  1. Find homo LDE general solution: $y_h = c_1y_1 + c_2y_2$
  2. Find Wronskian: $W = y_1y_2' - y_2y_1'$
  3. Find particular solution: $y_p = -y_1 \int \frac{y_2 r(x)}{W} dx + y_2 \int \frac{y_1 r(x)}{W} dx$
  4. General solution: $y = y_h + y_p$