

# ERG2011A Tutorial 7: Second-Order Linear Differential Equations

Prepared by Adrian Sai-wah TAM (swtam3@ie.cuhk.edu.hk)

8th November 2004

## 1 Previous knowledge

- Separable equations: (sometimes need substitution like  $u = y/x$  or  $v = ay + bx + k$ )

$$\begin{aligned}g(y)dy &= f(x)dx \\ \implies y &= G^{-1}(F(x) + C)\end{aligned}$$

- Exact differential equation:

$$M(x, y)dx + N(x, y)dy = 0 \quad \text{with} \quad \frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}$$

which solve by

$$\begin{aligned}\int M(x, y)dx &= f(x, y) + h(y) \\ \frac{\partial}{\partial y}f(x, y) + h'(y) &= N(x, y)\end{aligned}$$

- Use of integrating factors:

$$\begin{aligned}F(x, y)M(x, y)dx + F(x, y)N(x, y)dy &= 0 \\ \partial_y(F(x, y)M(x, y)) &= \partial_x(F(x, y)N(x, y))\end{aligned}$$

if  $F$  is single variable, it is:

$$\begin{aligned}F(x) &= \exp \int \frac{1}{N} \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) dx \\ F(y) &= \exp \int \frac{1}{M} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dy\end{aligned}$$

- Linear differential equation:

$$\begin{aligned}y' + p(x)y &= r(x) \\ \implies y(x) &= e^{-h} \left[ \int e^h r dx + C \right], \text{ where } h = \int p(x)dx\end{aligned}$$

If  $r(x) \equiv 0$ ,  $y(x) = Ce^{-h}$

- Bernoulli equation: (solve by substitution of  $u = y^{1-a}$ )

$$\begin{aligned}y' + p(x)y &= g(x)y^a \\ \implies u' + (1-a)p(x)u &= (1-a)g(x)\end{aligned}$$

## 2 Second-order Linear Equation

- Standard form of L.D.E.:

$$y'' + p(x)y' + q(x)y = r(x)$$

If  $r(x) \equiv 0$ , we call it homogeneous, otherwise it is non-homogeneous.  $p(x)$ ,  $q(x)$ ,  $r(x)$  can be anything independent of  $y$ .

- Be careful that usually, differential equation gives many solutions if we do not constrain it with initial values or boundary values

- Properties:

1. If  $y_1$  and  $y_2$  are solutions of  $y'' + p(x)y' + q(x)y = 0$ , then  $c_1y_1 + c_2y_2$  is also a solution  $\forall c_1, c_2 \in \mathbb{R}$

2. If  $y_1/y_2$  is not a real number, they are called independent solutions.

Then the general solutions of  $y'' + p(x)y' + q(x)y = 0$  is  $c_1y_1 + c_2y_2 \forall c_1, c_2 \in \mathbb{R}$ .

3. If  $y_3$  is a particular solution of  $y'' + p(x)y' + q(x)y = r(x)$ , then the general solution of it is  $c_1y_1 + c_2y_2 + y_3 \forall c_1, c_2 \in \mathbb{R}$ . In other words, the general solution of homogeneous equation plus any solution of non-homogeneous equation is the general solution of nonhomogeneous equation.

- A better way of determining independence of solutions is using the Wronski determinant:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$$

$W(y_1, y_2) \neq 0$  for some values of  $x$  if and only if  $y_1$  and  $y_2$  are linearly independent

– Reference: Book section 2.7

- Solving a homogeneous equation is therefore a critical part for general solution

## 3 Homogeneous Linear Equation

### 3.1 General coefficients

- Solving  $y'' + p(x)y' + q(x)y = 0$  generally can be difficult sometimes

– Hence we may find the first solution  $y_1$  by guessing and trials

- After  $y_1$  is found, we can find  $y_2$  using the method of reduction of order

1. Let  $y_2 = u(x, y)y_1$

2. Finally we find that:  $u(x, y) = \int U dx$ , where  $U = \frac{1}{y_1^2} \exp\left(-\int p(x) dx\right)$

– Reference: Book page 69-70

- If we found an homogeneous non-linear equation, we can convert it into linear equation sometimes:

–  $F(x, y', y'') = 0 \implies$  substitute  $z = y'$

–  $F(y, y', y'') = 0 \implies$  substitute  $z = y'$ ,  $y'' = z \frac{dz}{dy}$

- Example: Problem Set 2.1 Question 8:  $xy'' + y' = 0$

$$\begin{aligned}
 xy'' + y' &= 0 \\
 x &= -y'/y'' \\
 \frac{1}{x} &= -\frac{y''}{y'} \\
 \int \frac{dx}{x} &= -\int \frac{d(y')}{y'} \\
 \ln x &= -\ln(y') \\
 x &= \frac{1}{y'} \\
 y' &= \frac{1}{x} \\
 y &= \int \frac{dx}{x} \\
 &= \ln x
 \end{aligned}$$

Now we found  $y_1 = \ln x$ . Because the question is equivalent to  $y'' + y'/x = 0$ , we can have

$$\begin{aligned}
 U &= \frac{1}{y_1^2} \exp\left(-\int \frac{1}{x} dx\right) \\
 &= \frac{1}{(\ln x)^2} \exp(-\ln x) \\
 &= \frac{1}{x(\ln x)^2} \\
 u &= \int \frac{1}{x(\ln x)^2} dx \\
 &= \int \frac{1}{(\ln x)^2} d(\ln x) \\
 &= -\frac{1}{\ln x} \\
 y_2 &= uy_1 = -1
 \end{aligned}$$

Hence the general solution for  $xy'' + y' = 0$  is  $y = c_1 \ln x - c_2$  or  $y = c_1 \ln x + c_3$ . Alternatively, by taking care of the constants of integration, we can solve it in one-pass (just because we are lucky to have such simple equation):

$$\begin{aligned}
 xy'' + y' &= 0 \\
 x &= -y'/y'' \\
 \frac{1}{x} &= -\frac{y''}{y'} \\
 \int \frac{dx}{x} &= -\int \frac{d(y')}{y'} \\
 \ln x + C_1 &= -\ln(y') \\
 e^{C_1 x} &= \frac{1}{y'} \\
 y' &= e^{-C_1} \frac{1}{x} \\
 y &= e^{-C_1} \int \frac{dx}{x} \\
 &= e^{-C_1} \ln x + C_2 e^{-C_1} \\
 &= c_1 \ln x + c_2
 \end{aligned}$$

- Example: Problem Set 2.1 Question 10:  $y'' + (1 + y^{-1})y'^2 = 0$

$$\begin{aligned}
 y'' + (1 + \frac{1}{y})y'^2 &= 0 \\
 z \frac{dz}{dy} + (1 + \frac{1}{y})z^2 &= 0 \\
 yz \frac{dz}{dy} + (y + 1)z^2 &= 0 \\
 y \frac{dz}{dy} + (y + 1)z &= 0 \\
 \frac{y + 1}{y} dy &= -\frac{dz}{z} \\
 y + \ln y + C_0 &= \int (1 + \frac{1}{y}) dy = -\int \frac{dz}{z} = -\ln z \\
 C_1 y e^y &= \frac{1}{z} = \frac{dx}{dy} \\
 C_1 \int (y e^y) dy &= x \\
 C_1 (y - 1) e^y + C_2 &= x
 \end{aligned}$$

### 3.2 Constant coefficients

- If the equation is:  $y'' + ay' + by = 0$ , then:

1.  $y = e^{\lambda x}$  is a solution
2. Where  $\lambda$  satisfies the quadratic equation  $\lambda^2 + a\lambda + b = 0$   
we call this quadratic equation the characteristic equation

- If  $\Delta = a^2 - 4b$  is:

1.  $\Delta > 0$ , then  $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \quad \forall c_1, c_2 \in \mathbb{R}$
2.  $\Delta = 0$ , then  $y = (c_1 + c_2 x) e^{-ax/2} \quad \forall c_1, c_2 \in \mathbb{R}$ , whereas  $\lambda = -a/2$
3.  $\Delta < 0$ , then  $y = e^{-ax/2} (c_1 \cos \omega x + c_2 \sin \omega x) \quad \forall c_1, c_2 \in \mathbb{R}$ , whereas  $\omega = \sqrt{b - \frac{1}{4}a^2}$  and  $\lambda = -\frac{1}{2}a \pm i\omega$

- Example: Problem Set 2.2 Question 8:  $10y'' + 6y' - 4y = 0$

$$\begin{aligned}
 10y'' + 6y' - 4y &= 0 \\
 y'' + \frac{3}{5}y' - \frac{2}{5}y &= 0 \\
 \therefore \Delta &= \frac{9}{25} - 4 \left( \frac{-2}{5} \right) = \frac{17}{25} > 0 \\
 \text{Solving } 10\lambda^2 + 6\lambda - 4 &= 0 \\
 \text{Gives } \lambda &= -1 \text{ or } \frac{2}{5} \\
 \therefore y &= c_1 e^{-x} + c_2 e^{2x/5} \quad \forall c_1, c_2 \in \mathbb{R}
 \end{aligned}$$

- Example: Problem Set 2.2 Question 9:  $y'' + 2ky' + k^2y = 0$

$$\begin{aligned}
 y'' + 2ky' + k^2y &= 0 \\
 \implies \Delta &= (2k)^2 - 4(k^2) = 0 \\
 \therefore y &= (c_1 + c_2 x) e^{-2kx/2}
 \end{aligned}$$

- Example: Problem Set 2.3 Question 10:  $y'' - 2\sqrt{2}y' + 2.5y = 0$

$$\begin{aligned}
 y'' - 2\sqrt{2}y' + 2.5y &= 0 \\
 \implies \Delta &= 8 - 4(2.5) < 0 \\
 \therefore \omega &= \sqrt{2.5 - \frac{1}{4}(-2\sqrt{2})^2} \\
 &= \sqrt{2.5 - \frac{8}{4}} = \sqrt{0.5} \\
 &= \frac{1}{4} \\
 \therefore y &= e^{\sqrt{2}x} \left( c_1 \cos \frac{x}{4} + c_2 \sin \frac{x}{4} \right)
 \end{aligned}$$

- Knowing these technique can help you solve any cases in damping motions (see book section 2.5)

### 3.3 Partial constant coefficients — Euler-Cauchy Equations

- Euler-Cauchy equations have some of the coefficients are constant:

$$x^2y'' + axy' + by = 0$$

- A particular solution is given by:

1.  $y = x^m$

2. Where  $m$  satisfies the quadratic equation:  $m^2 + (a-1)m + b = 0$

- With different cases of the discriminants  $\Delta = (a-1)^2 - 4b$ ,

1.  $\Delta > 0$ , then  $y = c_1x^{m_1} + c_2x^{m_2}$

2.  $\Delta = 0$ , then  $y = (c_1 + c_2 \ln x)x^{(1-a)/2}$

3.  $\Delta < 0$ , then  $y = x^\mu [c_1 \cos(\nu \ln x) + c_2 \sin(\nu \ln x)]$ , where  $m = \mu \pm i\nu = (1-a) \pm i(4b - (a-1)^2)$ .

- Example: Problem Set 2.6 Question 8:  $(xD^2 + D)y = 0$

$$\begin{aligned}
 (xD^2 + D)y &= 0 \\
 xD^2y + Dy &= 0 \\
 xy'' + y' &= 0 \\
 x^2y'' + xy' + 0 \cdot y &= 0 \implies a = 1, b = 0 \\
 \therefore \Delta &= (1-1)^2 - 4(0) = 0 \\
 \therefore y &= (c_1 + c_2 \ln x)x^{(1-1)/2} \\
 &= c_1 + c_2 \ln x
 \end{aligned}$$

- Example: Problem Set 2.6 Question 10:  $(x^2D^2 + 0.7xD - 0.1)y = 0$

$$\begin{aligned}
 (x^2D^2 + 0.7xD - 0.1)y &= 0 \\
 x^2y'' + 0.7xy' - 0.1y &= 0 \\
 \therefore \Delta &= (0.7-1)^2 - 4(-0.1) > 0 \\
 \text{and } m &= 0.5 \text{ or } -0.2 \\
 \therefore y &= c_1x^{0.5} + c_2x^{-0.2}
 \end{aligned}$$

## 4 Non-Homogeneous Linear Equations

### 4.1 Constant coefficients

- Solving non-homogeneous equations can be difficult. So we look at a simpler case with constant coefficients:

$$y'' + ay' + by = r(x)$$

- Rule of thumb:  $y = y_h + y_p$   
(General solution is the composition of homogeneous general solution and a particular solution)
- How to find  $y_p$ ? We use the method of undetermined coefficients:

$r(x)$	$y_p$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n$ ( $n = 0, 1, \dots$ )	Polynomial: $\sum_{i=0}^n K_i x^i$
$k \cos \omega x$ or $k \sin \omega x$	$K \cos \omega x + M \sin \omega x$
$ke^{\alpha x} \cos \omega x$ or $ke^{\alpha x} \sin \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$

- Rules:
  1. If  $r(x)$  is as shown in the above table, choose the corresponding  $y_p$
  2. If  $y_p$  chosen is already represented by  $y_h$ , modify  $y_p$  by multiplying  $x$ ; or multiplying  $x^2$  if  $y_h$  is obtained with double root of  $\lambda$
  3. If  $r(x)$  is a sum of several functions above, choose  $y_p$  to be the sum of corresponding functions accordingly
- Example: Problem Set 2.8 Question 8

Verify  $y_p$  is a particular solution and find the general solution:  $(8D^2 - 6D + 1)y = 6 \cosh x$ ,  $y_p = \frac{1}{5}e^{-x} + e^x$

– Verification:

$$\begin{aligned}
 y_p &= \frac{1}{5}e^{-x} + e^x \\
 y_p' &= -\frac{1}{5}e^{-x} + e^x \\
 y_p'' &= \frac{1}{5}e^{-x} + e^x \\
 \therefore 8y_p'' - 6y_p' + y_p &= 8\left(\frac{1}{5}e^{-x} + e^x\right) - 6\left(-\frac{1}{5}e^{-x} + e^x\right) + \frac{1}{5}e^{-x} + e^x \\
 &= \frac{8+6+1}{5}e^{-x} + (8-6+1)e^x \\
 &= 3e^{-x} + 3e^x \\
 &= 6 \cdot \frac{e^{-x} + e^x}{2} = 6 \cosh x
 \end{aligned}$$

– General solution:

$$\begin{aligned}
 8y'' - 6y' + y &= 0 \\
 \implies m &= \frac{1}{2} \text{ or } \frac{1}{4} \\
 \therefore y_h &= c_1 e^{x/2} + c_2 e^{x/4} \\
 \therefore y &= c_1 e^{x/2} + c_2 e^{x/4} + \frac{1}{5}e^{-x} + e^x \quad \forall c_1, c_2 \in \mathbb{R}
 \end{aligned}$$

- Example: Problem Set 2.9 Question 8:  $y'' + 6y' + 9y = 50e^{-x} \cos x$

– Homogeneous solution,  $y_h$

$$\begin{aligned} y'' + 6y' + 9y &= 0 \\ \implies \Delta &= 6^2 - 4 \cdot 9 = 0 \\ \therefore y_h &= (c_1 + c_2x)e^{-3x} \quad \forall c_1, c_2 \in \mathbb{R} \end{aligned}$$

– Particular solution,  $y_p$

$$\begin{aligned} r(x) &= 50e^{-x} \cos x \\ \implies y_p &= e^{-x}(K \cos x + M \sin x) \\ \therefore y_p' &= -e^{-x}(K \cos x + M \sin x) + e^{-x}(-K \sin x + M \cos x) \\ y_p'' &= e^{-x}(K \cos x + M \sin x) - e^{-x}(-K \sin x + M \cos x) \\ &\quad - e^{-x}(-K \sin x + M \cos x) + e^{-x}(-K \cos x - M \sin x) \\ &= 2e^{-x}(K \sin x - M \cos x) \\ \therefore y_p'' + 6y_p' + 9y_p &= 2e^{-x}(K \sin x - M \cos x) \\ &\quad - 6e^{-x}(K \cos x + M \sin x) + 6e^{-x}(-K \sin x + M \cos x) \\ &\quad + 9e^{-x}(K \cos x + M \sin x) \\ &= 3e^{-x}(K \cos x + M \sin x) - 4e^{-x}(K \sin x - M \cos x) \\ &= (3K + 4M)e^{-x} \cos x + (3M - 4K)e^{-x} \sin x \\ \text{hence, } r(x) = 50e^{-x} \cos x &\implies \begin{cases} 3K + 4M = 50 \\ 3M - 4K = 0 \end{cases} \\ &\implies \begin{cases} K = \frac{25}{3} \\ M = \frac{25}{4} \end{cases} \\ \therefore y_p &= e^{-x} \left( \frac{25}{3} \cos x + \frac{25}{4} \sin x \right) \end{aligned}$$

– General solution:  $y_h + y_p$

$$y = (c_1 + c_2x)e^{-3x} + e^{-x} \left( \frac{25}{3} \cos x + \frac{25}{4} \sin x \right) \quad \forall c_1, c_2 \in \mathbb{R}$$

## 4.2 Method of variation of parameters

- If the coefficient is not constant, we solve it by the method of variation of parameters

– Caution: Complicated but almighty

- Definitions:

– Given a homogeneous counterpart of the equation, setting  $c_1 = 1, c_2 = 0$  and  $c_1 = 0, c_2 = 1$  respectively will give out  $y_1$  and  $y_2$ . We call them the basis

– Wronskian:  $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$

– Particular solution:

$$y_p = -y_1 \int \frac{y_2 r(x)}{W} dx + y_2 \int \frac{y_1 r(x)}{W} dx$$

- Example: Problem Set 2.10 Question 8:  $(D^2 + 6D + 9)y = 16e^{-3x}/(x^2 + 1)$

- $r(x)$  does not look like sum of things we know — cannot find  $y_p$  by looking up the table
- Solve homogeneous counterpart: as in Problem Set 2.9 Question 8

$$\begin{aligned}y_h &= (c_1 + c_2x)e^{-3x} \quad \forall c_1, c_2 \in \mathbb{R} \\ \therefore y_1 &= e^{-3x} \\ y_2 &= xe^{-3x}\end{aligned}$$

- Wronskian:

$$\begin{aligned}W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1' \\ &= e^{-3x} \cdot (e^{-3x} - 3xe^{-3x}) - xe^{-3x} \cdot (-3e^{-3x}) \\ &= e^{-6x} - 3xe^{-6x} + 3xe^{-6x} \\ &= e^{-6x}\end{aligned}$$

- Particular solution:

$$\begin{aligned}y_p &= -y_1 \int \frac{y_2 r(x)}{W} dx + y_2 \int \frac{y_1 r(x)}{W} dx \\ &= -e^{-3x} \int \frac{xe^{-3x} \cdot 16e^{-3x}/(x^2 + 1)}{e^{-6x}} dx + xe^{-3x} \int \frac{e^{-3x} \cdot 16e^{-3x}/(x^2 + 1)}{e^{-6x}} dx \\ &= -e^{-3x} \int \frac{16x}{x^2 + 1} dx + xe^{-3x} \int \frac{16}{x^2 + 1} dx \\ &= -16e^{-3x} \cdot \frac{1}{2} \ln(x^2 + 1) + 16xe^{-3x} \cdot \tan^{-1}(x) \\ &= e^{-3x} (16x \tan^{-1} x - 8 \ln(x^2 + 1))\end{aligned}$$

- General solution:

$$\begin{aligned}y &= y_h + y_p \\ &= (c_1 + c_2x)e^{-3x} + e^{-3x} (16x \tan^{-1} x - 8 \ln(x^2 + 1)) \\ &= e^{-3x} (c_1 + c_2x + 16x \tan^{-1} x - 8 \ln(x^2 + 1))\end{aligned}$$

- Example: Problem Set 2.10 Question 14:  $(x^2D^2 - 4xD + 6)y = 7x^4 \sin x$

- Solving homogeneous version: (this is an Euler-Cauchy Equation)

$$\begin{aligned}x^2y'' - 4xy' + 6y &= 0 \\ \implies \Delta &= (-4 - 1)^2 - 4(6) = 25 - 24 > 0 \\ \therefore m &= 2 \text{ or } 3 \\ \therefore y_h &= c_1x^2 + c_2x^3\end{aligned}$$

- Wronskian:

$$\begin{aligned}W &= y_1y_2' - y_2y_1' \\ &= x^2 \cdot (3x^2) - x \cdot (2x) \\ &= 3x^4 - 2x^4 \\ &= x^4\end{aligned}$$



– Particular solution:

$$\begin{aligned}
 y_p &= -y_1 \int \frac{y_2 r(x)}{W} dx + y_2 \int \frac{y_1 r(x)}{W} dx \\
 &= -x^2 \int \frac{x^3 \cdot 7x^4 \sin x}{x^4} dx + x^3 \int \frac{x^2 \cdot 7x^4 \sin x}{x^4} dx \\
 &= -7x^2 \int x^3 \sin x dx + 7x^3 \int x^2 \sin x dx \quad \leftarrow \text{solve this using "integration by part"} \\
 &= -7x^2 ((6x - x^3) \cos x + (3x^2 - 6) \sin x) + 7x^3 ((2 - x^2) \cos x + 2x \sin x) \\
 &= -7x^2 ((-x^3 - x^2 + 6x + 2) \cos x + (3x^2 + 2x - 6) \sin x)
 \end{aligned}$$

– General solution:

$$\begin{aligned}
 y &= y_h + y_p \\
 &= c_1 x^2 + c_2 x^3 - 7x^2 ((-x^3 - x^2 + 6x + 2) \cos x + (3x^2 + 2x - 6) \sin x)
 \end{aligned}$$

– How to use integration by part to solve  $\int x^2 \sin x dx$  and  $\int x^3 \sin x dx$ ? — in case you forgot about that

$$\begin{aligned}
 \int x^2 \sin x dx &= - \int x^2 d(\cos x) \\
 &= -x^2 \cos x - (- \int \cos x d(x^2)) \\
 &= -x^2 \cos x + 2 \int x \cos x dx \\
 &= -x^2 \cos x + 2 \int x d(\sin x) \\
 &= -x^2 \cos x + 2x \sin x - 2 \int \sin x dx \\
 &= -x^2 \cos x + 2x \sin x + 2 \cos x
 \end{aligned}$$

$$\begin{aligned}
 \int x^3 \sin x dx &= - \int x^3 d(\cos x) \\
 &= -x^3 \cos x - (- \int \cos x d(x^3)) \\
 &= -x^3 \cos x + 3 \int x^2 \cos x dx \\
 &= -x^3 \cos x + 3 \int x^2 d(\sin x) \\
 &= -x^3 \cos x + 3x^2 \sin x - 3 \int \sin x d(x^2) \\
 &= -x^3 \cos x + 3x^2 \sin x - 6 \int x \sin x dx \\
 &= -x^3 \cos x + 3x^2 \sin x + 6 \int x d(\cos x) \\
 &= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \int \cos x dx \\
 &= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x
 \end{aligned}$$

## 5 Higher-order Linear Equations with Constant Coefficients

- Standard form:

$$\frac{d^n y}{dx^n} + p_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_1(x) \frac{dy}{dx} + p_0(x)y = r(x)$$

- The Wronskian in Second-order linear differential equation is a  $2 \times 2$  matrix, but here, it is a  $n \times n$  matrix:

$$W(y_1, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

- We don't deal with the cases that coefficients of  $\frac{d^n y}{dx^n}$  are not a real number in this course, i.e., we only handle

$$y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_1y' + p_0y = r(x)$$

### 5.1 Homogeneous Equations

- Similar to second-order cases, we have the characteristic equation:  $\lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_1\lambda + p_0 = 0$ 
  - Roots of the characteristic equation are  $\lambda_1, \lambda_2, \dots, \lambda_n$
  - Order- $n$  equations have  $n$  basis in any cases
- If all  $n$  roots are distinct, we have  $n$  basis:  $y_k = e^{\lambda_k x}$ ,  $k = 1, \dots, n$
- If some of the roots are repeated  $m$  times, we have  $m$  of the  $n$  basis in the form:  $y_k = x^k e^{\lambda x}$ ,  $k = 0, \dots, m - 1$
- If some the roots are a pair of complex conjugates,  $\lambda = \gamma \pm i\omega$ , we have 2 of the  $n$  basis in the form:  $y_1 = e^{\gamma x} \cos \omega x$ ,  $y_2 = e^{\gamma x} \sin \omega x$
- If some the roots are repeated complex conjugates repeated  $m$  times, we have  $2m$  of the  $n$  basis in the form  $y_k = x^k e^{\gamma x} \cos \omega x$ ,  $y_{m+k} = x^k e^{\gamma x} \sin \omega x$ ,  $k = 0, \dots, m - 1$

### 5.2 Non-Homogeneous Equations

- Follow the method as in second-order, except the general solution for homogeneous counterpart,  $y_h$ , is consisting of  $n$  components and  $n$  arbitrary constants
- $y = y_h + y_p = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n + y_p$
- Easy, no magic!

## 6 Summary of Second-order Linear D.E.

- Standard form of L.D.E.:  $y'' + p(x)y' + q(x)y = r(x)$
- Properties of L.D.E.:
  1. If  $y_1$  and  $y_2$  are solutions of the homo LDE and  $y_1/y_2 \neq \text{constant}$ , they are independent solutions
  2. If  $y_1$  and  $y_2$  are independent solutions of the homo LDE, then  $c_1y_1 + c_2y_2$  is the general solution
  3. If  $y_3$  is a particular solution of non-homo LDE, then the general solution of it is  $c_1y_1 + c_2y_2 + y_3$
- Wronski determinant:  $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$ 
  - $W(y_1, y_2) \neq 0 \iff y_1$  and  $y_2$  are linearly independent
- If  $y_1$  is one solution of a homo LDE, then  $y_2 = y_1 \int \frac{1}{y_1^2} e^{-\int p(x)dx} dx$  (method of reduction of order)
- Substitution for conversion to homo LDE:
  - $F(x, y', y'') = 0 \implies$  substitute  $z = y'$
  - $F(y, y', y'') = 0 \implies$  substitute  $z = y', y'' = z \frac{dz}{dy}$
- Special homo LDE:

	Constant coefficient	Eular-Cauchy equation
	$y'' + ay' + by = 0$	$x^2y'' + axy' + by = 0$
Char. eqn	$\lambda^2 + a\lambda + b = 0$	$m^2 + (a-1)m + b = 0$
$\Delta$	$a^2 - 4b$	$(a-1)^2 - 4b$
$\Delta > 0$	$y = c_1e^{\lambda_1x} + c_2e^{\lambda_2x}$	$y = c_1x^{m_1} + c_2x^{m_2}$
$\Delta = 0$	$y = (c_1 + c_2x)e^{-ax/2}$	$y = (c_1 + c_2 \ln x)x^{(1-a)/2}$
$\Delta < 0$	$\lambda = -\frac{1}{2}a \pm i\omega = -\frac{1}{2}a \pm i\sqrt{b - \frac{1}{4}a^2}$ $y = e^{-ax/2}(c_1 \cos \omega x + c_2 \sin \omega x)$	$m = \mu \pm i\nu = (1-a) \pm i(4b - (a-1)^2)$ $y = x^\mu [c_1 \cos(\nu \ln x) + c_2 \sin(\nu \ln x)]$

- Non-homo L.D.E.: Guess  $y_p$  and then solve for the unknown coefficients:
  1. If  $r(x)$  is as shown in the table, choose the corresponding  $y_p$
  2. If  $y_p$  chosen is already represented by  $y_h$ , modify  $y_p$  by multiplying  $x$ ; or multiplying  $x^2$  if  $y_h$  is obtained with double root of  $\lambda$
  3. If  $r(x)$  is a sum of several functions above, choose  $y_p$  to be the sum of corresponding functions accordingly

$r(x)$	$y_p$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n$ ( $n = 0, 1, \dots$ )	Polynomial: $\sum_{i=0}^n K_i x^i$
$k \cos \omega x$ or $k \sin \omega x$	$K \cos \omega x + M \sin \omega x$
$ke^{\alpha x} \cos \omega x$ or $ke^{\alpha x} \sin \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$

- Method of variation of parameters: for use when  $r(x)$  is not in the above table
  1. Find homo LDE general solution:  $y_h = c_1y_1 + c_2y_2$
  2. Find Wronskian:  $W = y_1y_2' - y_2y_1'$
  3. Find particular solution:  $y_p = -y_1 \int \frac{y_2 r(x)}{W} dx + y_2 \int \frac{y_1 r(x)}{W} dx$
  4. General solution:  $y = y_h + y_p$