ERG2011A Tutorial 10: Fourier Series and Fourier Transform

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1 Summary of Fourier Series in Tutorial 9

- Objective: Convert a periodic function as sum of sine and cosine functions
  - Every sine and consine functions are in harmonics of the original function (i.e. period of each component is an integer multiple of the period)

Fourier representation with period 2π	Fourier representation with period $p = 2L$
$f(x) = a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$	$f(x) = a_0 + \sum_{k=1}^{\infty}\left(a_k \cos \frac{k\pi}{L}x + b_k \sin \frac{k\pi}{L}x\right)$
where: $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$	where: $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$
$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx dx$	$a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos \frac{k\pi}{L}x dx$
$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx dx$	$b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin \frac{k\pi}{L}x dx$

- Even function means $f(-x) = f(x)$; odd function means $f(-x) = -f(x)$. Examples are cosine and sine.

Even function	Odd function
$f(-x) = f(x)$	$f(-x) = -f(x)$
Example: $\cos x$	Example: $\sin x$
$\int_{-L}^{L} f_{\text{even}}(x)dx = 2\int_{0}^{L} f_{\text{even}}(x)dx$	$\int_{-L}^{L} f_{\text{odd}}(x)dx = 0$
$f_{\text{even}}(x) = a_0 + \sum_{k=1}^{\infty}\left(a_k \cos \frac{k\pi}{K}x\right)$	$f_{\text{odd}}(x) = \sum_{k=1}^{\infty}\left(b_k \sin \frac{k\pi}{K}x\right)$

- Further properties of Fourier series representation:
  - $f(x) = f_1(x) + f_2(x)$ then the Fourier series is the sum of every corresponding coefficients
  - $cf(x)$ has the Fourier series with each Fourier coefficients of $f(x)$ multiplied by c

2 Complex Fourier Series

2.1 Euler's equation

- As you learnt in high school:

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{i\pi}+1 &= 0 \quad \longleftarrow \text{Euler's equation, contains all important symbols in math: } e, \pi, i, 1, 0 \end{aligned}$$

- Therefore, we have:

$$\begin{aligned} \cos x &= \frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right) \\ \sin x &= \frac{1}{2}\left(e^{i\theta}-e^{-i\theta}\right) \end{aligned}$$

2.2 Fourier series with e^{inx}

- For functions with period 2π,

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{inx} \\ c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \end{aligned}$$

  - Compare with the $\sin nx$ and $\cos nx$ version,

$$\begin{aligned} c_0 &= a_0 \\ c_n &= \frac{a_n - ib_n}{2} \qquad (n>0) \\ c_{-n} &= \frac{a_n + ib_n}{2} \qquad (n>0) \end{aligned}$$

- But if the function is with period $p=2L$,

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \\ c_n &= \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx \end{aligned}$$

- Example: Problem Set 10.5 Question 5. Find the complex Fourier series of $f(x)=x \quad (0<x<2\pi)$

$$\begin{aligned} f(x) &= \begin{cases} x & 0<x<\pi \\ 2\pi + x & -\pi<x<0 \end{cases} \\ \therefore \quad c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \\ &= \frac{1}{2\pi}\left(\int_{-\pi}^{0}(2\pi+x)e^{-inx}dx \quad + \int_{0}^{\pi} xe^{-inx}dx\right) \\ &= \frac{1}{2\pi}\left(\left[\frac{1}{-in}(2\pi+x)e^{-inx}\right]_{-\pi}^{0} - \frac{1}{-in}\int_{-\pi}^{0} e^{-inx}dx \quad + \left[\frac{1}{-in}xe^{-inx}\right]_{0}^{\pi} - \frac{1}{-in}\int_{0}^{\pi} e^{-inx}dx\right) \\ &= \frac{1}{2\pi}\left(\left[\frac{1}{-in}(2\pi+x)e^{-inx}\right]_{-\pi}^{0} - \frac{1}{-in}\left[\frac{1}{-in}e^{-inx}\right]_{-\pi}^{0} \quad + \left[\frac{1}{-in}xe^{-inx}\right]_{0}^{\pi} - \frac{1}{-in}\left[\frac{1}{-in}e^{-inx}\right]_{0}^{\pi}\right) \end{aligned}$$

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\left(\left[\frac{1}{-in}(2\pi) - \frac{1}{-in}(\pi)e^{-in\pi}\right] - \frac{1}{-in}\left[\frac{1}{-in} - \frac{1}{-in}e^{in\pi}\right] + \left[\frac{1}{-in}\pi e^{-in\pi}\right] - \frac{1}{-in}\left[\frac{1}{-in}e^{-in\pi} - \frac{1}{-in}\right]\right) \\
&= \frac{1}{2\pi}\left(\left[\frac{1}{-in}(2\pi)\right] - \frac{1}{-in}\left[-\frac{1}{-in}e^{in\pi}\right] - \frac{1}{-in}\left[\frac{1}{-in}e^{-in\pi}\right]\right) \\
&= \frac{1}{2\pi}\left(-\frac{2\pi}{in} - \frac{1}{n^2}e^{in\pi} + \frac{1}{n^2}e^{-in\pi}\right) \\
&= \frac{1}{2\pi}\left(-\frac{2\pi}{in} + \frac{1}{n^2}e^{in\pi}(-1 + e^{-2in\pi})\right) \qquad \longleftarrow \text{care of this trick!} \\
&= \frac{1}{2\pi}\left(-\frac{2\pi}{in} + \frac{1}{n^2}e^{in\pi}(-1 + 1)\right) \\
&= \frac{1}{2\pi}\left(-\frac{2\pi}{in}\right) \\
&= \frac{-1}{in} \\
&= \frac{i}{n} \qquad (n \neq 0) \\
c_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{0}(2\pi + x)dx + \frac{1}{2\pi}\int_{0}^{\pi} xdx \\
&= \frac{1}{2\pi}\int_{-\pi}^{0} 2\pi dx + \frac{1}{2\pi}\int_{-\pi}^{0} xdx + \frac{1}{2\pi}\int_{0}^{\pi} xdx \\
&= \frac{1}{2\pi}\int_{-\pi}^{0} 2\pi dx + \frac{1}{2\pi}\int_{-\pi}^{\pi} xdx \qquad \longleftarrow \text{Odd function!} \\
&= \frac{1}{2\pi}\int_{-\pi}^{0} 2\pi dx \\
&= 0 - (-\pi) \\
&= \pi \\
\therefore \quad f(x) &= \pi + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{i}{n}e^{inx}
\end{aligned}$$

Actually, the integration can be done in any complete period instead of $-\pi < x < \pi$. For example:

$$\begin{aligned}
f(x) &= x \qquad (0 < x < 2\pi) \\
\therefore \quad c_n &= \frac{1}{2\pi}\int_{0}^{2\pi} f(x)e^{-inx}dx \\
&= \frac{1}{2\pi}\left(\int_{0}^{2\pi} xe^{-inx}dx\right) \\
&= \frac{1}{2\pi}\left(\left[\frac{1}{-in}xe^{-inx}\right]_0^{2\pi} - \frac{1}{-in}\int_{0}^{2\pi} e^{-inx}dx\right) \\
&= \frac{1}{2\pi}\left(\left[\frac{1}{-in}2\pi e^{-2in\pi} - 0\right] - \frac{1}{-in}\left[\frac{1}{-in}e^{-inx}\right]_0^{2\pi}\right) \\
&= \frac{1}{2\pi}\left(\frac{1}{-in}2\pi e^{-2in\pi} - \frac{1}{-n^2}(e^{-2in\pi} - 1)\right) \\
&= \frac{1}{2\pi}\left(\frac{1}{-in}2\pi e^{-2in\pi} - \frac{1}{-n^2}(1 - 1)\right) \\
&= \frac{1}{-in}e^{-2in\pi} \\
&= \frac{1}{-in} \\
&= \frac{i}{n} \qquad (n \neq 0)
\end{aligned}$$

$$
\begin{aligned}
c_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} xdx \\
&= \frac{1}{2\pi}\left[\frac{1}{2}x^2\right]_0^{2\pi} \\
&= \frac{4\pi^2}{4\pi} \\
&= \pi \\
\therefore \quad f(x) &= \pi + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{i}{n}e^{inx}
\end{aligned}
$$

In addition, if we want to write these into real number terms, we can derive as follows:

$$
\begin{aligned}
f(x) &= \pi + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{i}{n}e^{inx} \\
&= \pi + \sum_{n=1}^{\infty} \frac{i}{n}e^{inx} - \sum_{n=1}^{\infty} \frac{i}{n}e^{-inx} \\
&= \pi + \sum_{n=1}^{\infty} \frac{i}{n}(e^{inx} - e^{-inx}) \\
&= \pi + \sum_{n=1}^{\infty} \frac{i}{n}(2i\sin nx) \\
&= \pi + \sum_{n=1}^{\infty} \frac{-2}{n}\sin nx \\
&= \pi - 2\sum_{n=1}^{\infty} \frac{1}{n}\sin nx
\end{aligned}
$$

3 Approximation by Trigonometric Polynomials

3.1 Parseval's relation (for proof, see lecture note P.11)

$$
\begin{aligned}
f(x) &= a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \\
\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx &= 2a_0^2 + \sum_{n+1}^{\infty}(a_n^2 + b_n^2)
\end{aligned}
$$

3.2 Approximation

- In Fourier series representation, if we sums the terms up to $n = N$ instead of $n = \infty$, we get an trigonometric polynomial of degree N, which is an approximation of the periodic function:

$$
f(x) \approx F(x) = a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx)
$$

- The error between $f(x)$ and $F(x)$ is represented by "square error":

$$\begin{aligned} E &= \int_{-\pi}^{\pi} (f-F)^2 dx \\ &= \int_{-\pi}^{\pi} f^2(x)dx - \pi\left[2a_0^2 + \sum_{n=1}^{N}(a_n^2+b_n^2)\right] \end{aligned}$$

  - Derivation please see lecture note P.12 or book P.555.
  - Actually, we can have $F(x) = A_0 + \sum_{n=1}^{N}(A_n \cos nx + B_n \sin nx)$ with any other coefficients A_n, B_n. But it can be proved that, E is minimum only if $A_n = a_n$ and $B_n = b_n$. That is, they are Fourier coefficients of $f(x)$.
  - The square error, E, by definition is non-negative, i.e. we have the Bessel inequality:

$$2a_0^2 + \sum_{n=1}^{N}(a_n^2+b_n^2) \leq \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx$$

    and taking the limit for $N \to \infty$, the inequality will become Parseval's theorem.

4 Fourier Transform

4.1 Formulae of Transform

- What if your function is not periodic?
- Answer: Assume it is periodic. For example, $f(x) = x$, repeats as if $-L < x < L$. Then take the limit on $L \to \infty$.
- We call this the Fourier Transform. Which is the extension on the Fourier Series to cover those non-periodic functions
  - Now, any integrable function is a sum of (may be infinitely many) trigonometric functions
- Fourier transform: $F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx = \mathcal{F}\{f(x)\}$
- Inverse Fourier transform: $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}d\omega = \mathcal{F}^{-1}\{F(\omega)\}$
  - Hint: Mathematica has those function

4.2 Meaning of Fourier Transform

- Assuming that all functions is a summation of trigonometric functions
- Hence all functions has some "frequencies" behind it, but with different magnitude
- We are going to present how each frequency is different from the other frequency (ω) in magnitude, but how the function's magnitude change with respect to time
- Example: Human can hear 20Hz to 20kHz only. In MP3 compression, the first step is to do a Fourier transform to the sound wave (magnitude vs time) so that we get the composition of each frequency component. Then, strip off those frequency out of 20Hz to 20kHz. Then compress the sound. Therefore, MP3 is lossy compression on sound data — and uses Fourier transform to choose what to loss!

4.3 Fourier Transformation Properties

- Basically, Fourier transform is identical to Laplace transform, by replacing $s = i\omega$, i.e., assuming pure imaginary s.
- Hence they share a lot of properties,
  - Linearity: $af(x) + bg(x) \overset{\mathcal{F}}{\Longleftrightarrow} aF(\omega) + bG(\omega)$
  - Derivative: $f'(x) \overset{\mathcal{F}}{\Longleftrightarrow} i\omega F(\omega)$
  - Second derivative: $f''(x) \overset{\mathcal{F}}{\Longleftrightarrow} -\omega^2 F(\omega)$
  - Convolution: $f(x) * g(x) \overset{\mathcal{F}}{\Longleftrightarrow} \sqrt{2\pi}F(\omega)G(\omega)$