ERG2011A Ultimate Tutorial: Whole Semester in a Nutshell

Prepared by Adrian Sai-wah TAM (swtam3@ie.cuhk.edu.hk)

6th December 2004

1 Summary of Course

You have learnt:

- Vector operation
- Vector differentiation (grad, div, curl)
- Vector integration (Green's, Stroke's, GDT)
- Differential equation (homogeneous, non-homo., higher-order)
- Laplace transform
- Fourier series, and some transform

2 Vectors in a Nutshell

2.1 Vector operation

- Vector in ordered triple notation: $\vec{x} = [x_1, x_2, x_3]$
- Vector dot product: $\vec{x} \cdot \vec{y} = |\vec{x}||\vec{y}| \cos\theta = x_1y_1 + x_2y_2 + x_3y_3$
- Vector cross product: $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$
- Vector triple product: $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$

2.2 Vector differentiation

- 3D vector function: $\mathbf{v}(t) = [v_1(t), v_2(t), v_3(t)]$
- Comparing vector differentiation and scalar differentation:

Vector Differentiation	Scalar Differentiation
$\frac{d}{dt}c\mathbf{v}(t) = c\frac{d}{dt}\mathbf{v}(t)$	$\frac{d}{dx}cf(x) = c\frac{d}{dx}f(x)$
$\frac{d}{dt}[\mathbf{u}(t)+\mathbf{v}(t)] = \frac{d\mathbf{u}(t)}{dt} + \frac{d\mathbf{v}(t)}{dt}$	$\frac{d}{dx}[f(x)+g(x)] = \frac{df(x)}{dx} + \frac{dg(x)}{dx}$
$\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}(t)\cdot\frac{d\mathbf{v}(t)}{dt} + \frac{d\mathbf{u}(t)}{dt}\cdot\mathbf{v}(t)$ $\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}(t)\times\frac{d\mathbf{v}(t)}{dt} + \frac{d\mathbf{u}(t)}{dt}\times\mathbf{v}(t)$	$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{dg(x)}{dx} + \frac{df(x)}{dx}g(x)$
$\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)\times\mathbf{w}(t)] = \frac{d\mathbf{u}(t)}{dt}\cdot\mathbf{v}(t)\times\mathbf{w}(t) + \mathbf{u}(t)\cdot\frac{d\mathbf{v}(t)}{dt}\times\mathbf{w}(t) + \mathbf{u}(t)\cdot\mathbf{v}(t)\times\frac{d\mathbf{w}(t)}{dt}$	
$\mathbf{v}'(t) = [v_x(t), v_y(t), v_z(t)]' = v_x'(t)\mathbf{i} + v_y'(t)\mathbf{j} + v_z'(t)\mathbf{k}$	

- The tangent of curve $\mathbf{r}(t)$ at the point $\mathbf{r}(\tau)$ is: $\mathbf{s}(t) = \mathbf{r}(\tau) + t\mathbf{r}'(\tau)$
- Length of curve $\mathbf{r}(t)$ from point $\mathbf{r}(a)$ to $\mathbf{r}(b)$ is: $\ell = \int_a^b \sqrt{\mathbf{r}'(t)\cdot\mathbf{r}'(t)}dt$
- Vector differential operator: Nabla, $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$

	grad	div	curl	Laplacian
Notation	grad $f = \nabla f$ $\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$	div $\mathbf{v} = \nabla\cdot\mathbf{v}$ $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$	curl $\mathbf{v} = \nabla\times\mathbf{v}$ $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$	$\nabla^2 f =$ div grad f $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla\cdot(\nabla f)$
Value	Vector	Scalar	Vector	Scalar

- Directional derivative: $D_{\hat{\mathbf{a}}}f = (\text{grad } f)\cdot\hat{\mathbf{a}}$ (slope of f at the direction of $\hat{\mathbf{a}}$)

2.3 Vector integration

2.3.1 Line integral

- Line integral: $\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\frac{d\mathbf{r}}{dt}dt = \int_a^b (F_1dx + F_2dy + F_3dz) = \int_a^b [F_1x'(t) + F_2y'(t) + F_3z'(t)]\,dt$
  - Integrate for $\mathbf{r}$ sweeping C, but we represent $\mathbf{r}$ as a function of t, and C is defined by $\mathbf{r}(t) = [x(t), y(t), z(t)]$ for $t = a$ to $t = b$
  - Line integral *may* depend on the actual path of C
- Line independent integral:
  - Thm 1 (potential energy): $\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r}$ is path independent iff we can find a function f in such that $\mathbf{F} = \text{grad}\, f$
  - Thm 2: $\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r}$ is path independent iff $\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = 0$ for all closed path C
  - Thm 3 (Exact differential): $\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r}$ is path independent iff $\text{curl}\,\mathbf{F} = \mathbf{0}$
    * Exact: $F_1dx + F_2dy + F_3dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$, or equiv. $\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}$, $\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$, $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$
- If the line integral is path-independent, we have $\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} = f(\mathbf{b}) - f(\mathbf{a})$
- Double Integral: Integrating over an area R, $\iint_R f(x,y)dxdy = \int_a^b \left[\int_{g(x)}^{h(x)} f(x,y)dy\right]dx = \int_c^d \left[\int_{p(y)}^{q(y)} f(x,y)dx\right]dy$

- Change of variable in double integral: $\iint_R f(x,y)dxdy = \iint_{R'} f(x(u,v),y(u,v)) \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} dudv$

2.3.2 Green's theorem

- Green's Theorem: $\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \oint_C (F_1 dx + F_2 dy)$
  - R is a closed bounded region in the xy-plane and its boundary is C
  - Alternative form: $\iint_R (\text{curl}\,\mathbf{F}) \cdot \mathbf{k} dxdy = \oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$
  - Counterclockwise is positive
- Finding Cartesian area using Green's theorem: $A = \frac{1}{2}\oint_C (xdy - ydx)$
- Finding polar area using Green's theorem: $A = \frac{1}{2}\oint_C r^2 d\theta$

2.3.3 Surface integrals

- Parametric form of curve (has one variable): $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$
- Parametric form of surface (has two variables): $\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$
  - A curve on the surface: Relating u and v: $\tilde{\mathbf{r}}(t) = \mathbf{r}(u(t), v(t))$
  - Tangents of this curve: $\tilde{\mathbf{r}}'(t) = \frac{d\tilde{\mathbf{r}}}{dt} = \frac{\partial \mathbf{r}}{\partial u}u' + \frac{\partial \mathbf{r}}{\partial v}v' = \mathbf{r}_u u' + \mathbf{r}_v v'$
  - Tangent plane: $h\frac{\partial \mathbf{r}}{\partial u} + k\frac{\partial \mathbf{r}}{\partial v}$ and unit normal is: $\mathbf{n} = \hat{\mathbf{N}} = \frac{1}{|\mathbf{r}_u \times \mathbf{r}_v|}\mathbf{r}_u \times \mathbf{r}_v$
  - If the surface S is represented by $g(x,y,z) = 0$, then: $\mathbf{n} = \frac{1}{|\text{grad}\, g|}\text{grad}\, g$

2.3.4 Flux integral

- The flux (mass of fluid per unit time) across a surface: $\iint_S \mathbf{F} \cdot \mathbf{n} dA = \iint_R \mathbf{F}[\mathbf{r}(u,v)] \cdot \mathbf{N}(u,v) dudv$
  - If $\begin{cases} \mathbf{F} &= F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} \\ \mathbf{n} &= \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k} \\ N &= N_1\mathbf{i} + N_2\mathbf{j} + N_3\mathbf{k} \end{cases}$, then $\begin{cases} \iint_S \mathbf{F} \cdot \mathbf{n} dA &= \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma) dA \\ \iint_S \mathbf{F} \cdot \mathbf{n} dA &= \iint_R (F_1N_1 + F_2N_2 + F_3N_3) dudv \\ \iint_S \mathbf{F} \cdot \mathbf{n} dA &= \iint_S (F_1 dydz + F_2 dzdx + F_3 dxdy) \end{cases}$
- Surface integral without regard to direction: $\iint_S G(\mathbf{r}) dA = \iint_R G(\mathbf{r}(u,v)) |\mathbf{N}(u,v)| dudv$
  - $\iint_S G(\mathbf{r}) dA = \iint_{R'} G(x,y,f(x,y)) \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dxdy$

2.3.5 Gauss' Divergence Theorem

1. $\iiint_T \nabla \cdot \mathbf{F} dV = \iint_S \mathbf{F} \cdot \mathbf{n} dA$
2. $\iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dxdydz = \iint_S (F_1 dydz + F_2 dzdx + F_3 dxdy)$

2.3.6 Stroke's Theorem

- For a particle rotating along an axis with the locus of rotation has radius $\mathbf{r}$, rotating with angular velocity ω and instantaneous velocity $\mathbf{v}$, then $\omega \times \mathbf{r} = \mathbf{v}$.
  - $\nabla \times \mathbf{v} = \frac{1}{2}\omega$
- Stroke's theorem: $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dx dy = \oint_C \mathbf{F} \cdot d\mathbf{r}$

3 Differential Equations in a Nutshell

3.1 Simple types:

- Separable equations:

$$
\begin{aligned}
g(y)dy &= f(x)dx \\
\int g(y)dy &= \int f(x)dx \\
G(y) &= F(x) + C \\
y &= G^{-1}(F(x) + C)
\end{aligned}
$$

- Exact differential equation:
  - Criteria 1: Differential equation looks like: $M(x,y)dx + N(x,y)dy = 0$
  - Criteria 2: M and N are complementary partial derivatives, i.e. $\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$
  - Solve by finding $u(x,y) = \int M(x,y)dx = \int N(x,y)dy$
- Integrating factors: Find a function $F(x,y)$ such that $F(x,y)M(x,y)dx + F(x,y)N(x,y)dy = 0$, i.e. the equation becomes exact
  - For simplicity, we usually assume $F(x)$ or $F(y)$ only, i.e. single-variable factors, and they are:

$$
\begin{aligned}
F(x) &= \exp \int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) dx \\
F(y) &= \exp \int \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy
\end{aligned}
$$

3.2 Linear differential equations

- Standard form: $y' + p(x)y = r(x)$. The solution is $y(x) = e^{-h}\left[\int e^h r dx + C\right]$, where $h = \int p(x)dx$
- If $r(x) \equiv 0$, it is called homogeneous, and then $y(x) = Ce^{-h}$
- Bernoulli equations is those looks like:$y' + p(x)y = g(x)y^a$
  - Solution: By substitution of $u = y^{1-a}$, we can convert the above equation into $u' + (1-a)p(x)u = (1-a)g(x)$

3.3 Second/Higer order linear differential equations

- Standard form of 2nd-order L.D.E.: $y'' + p(x)y' + q(x)y = r(x)$
- Properties:
  1. If y_1 and y_2 are solutions of the homo LDE and $y_1/y_2 \neq$constant, they are independent solutions
  2. If y_1 and y_2 are independent solutions of the homo LDE, then $c_1y_1 + c_2y_2$ is the general solution
  3. If y_3 is a particular solution of non-homo LDE, then the general solution of it is $c_1y_1 + c_2y_2 + y_3$
- Wronski determinant:$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$
  - $W(y_1, y_2) \not\equiv 0 \iff y_1$ and y_2 are linearly independent
- If y_1 is one solution of a homo LDE, then $y_2 = y_1 \int \frac{1}{y_1^2} e^{-\int p(x)dx} dx$ (method of reduction of order)
- Substitution for conversion to homo LDE:
  - $F(x, y', y'') = 0 \implies$ substitute $z = y'$
  - $F(y, y', y'') = 0 \implies$ substitute $z = y'$, $y'' = z\frac{dz}{dy}$
- Special homo LDE:

	Constant coefficient	Eular-Cauchy equation
	$y'' + ay' + by = 0$	$x^2y'' + axy' + by = 0$
Char. eqn	$\lambda^2 + a\lambda + b = 0$	$m^2 + (a-1)m + b = 0$
Δ	$a^2 - 4b$	$(a-1)^2 - 4b$
$\Delta > 0$	$y = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x}$	$y = c_1x^{m_1} + c_2x^{m_2}$
$\Delta = 0$	$y = (c_1 + c_2x)e^{-ax/2}$	$y = (c_1 + c_2 \ln x)x^{(1-a)/2}$
$\Delta < 0$	$\lambda = -\frac{1}{2}a \pm i\omega = -\frac{1}{2}a \pm i\sqrt{b - \frac{1}{4}a^2}$ $y = e^{-ax/2}(c_1 \cos\omega x + c_2 \sin\omega x)$	$m = \mu \pm i\nu = (1-a) \pm i\left(4b - (a-1)^2\right)$ $y = x^\mu\left[c_1 \cos(\nu \ln x) + c_2 \sin(\nu \ln x)\right]$

- Non-homo L.D.E.: Guess y_p and then solve for the unknown coefficients:
  1. If $r(x)$ is as shown in the table, choose the corresponding y_p
  2. If y_p chosen is already represented by y_h, modify y_p by multiplying x; or multiplying x^2 if y_h is obtained with double root of λ
  3. If $r(x)$ is a sum of several functions above, choose $y_p$to be the sum of corresponding functions accordingly

$r(x)$	y_p
$ke^{\gamma x}$	$Ce^{\gamma x}$
kx^n $(n = 0, 1, \ldots)$	Polynomial: $\sum_{i=0}^{n} K_ix^i$
$k\cos\omega x$ or $k\sin\omega x$	$K\cos\omega x + M\sin\omega x$
$ke^{\alpha x}\cos\omega x$ or $ke^{\alpha x}\sin\omega x$	$e^{\alpha x}(K\cos\omega x + M\sin\omega x)$

- Method of variation of parameters: for use when $r(x)$ is not in the above table
  1. Find homo LDE general solution: $y_h = c_1y_1 + c_2y_2$
  2. Find Wronskian: $W = y_1y_2' - y_2y_1'$
  3. Find particular solution: $y_p = -y_1 \int \frac{y_2r(x)}{W}dx + y_2 \int \frac{y_1r(x)}{W}dx$
  4. General solution: $y = y_h + y_p$

4 Laplace Transform in a Nutshell

$f(t)$	$F(s)$
$f(t)$	$\int_0^\infty e^{-st}f(t)dt$
$af(t)+bg(t)$	$aF(s)+bG(s)$
$e^{at}f(t)$	$F(s-a)$
$f(t-a)u(t-a)$	$e^{-as}F(s)$
$f(at)$	$\frac{1}{a}F(\frac{s}{a})$
$f^{(n)}(t)$	$s^nF(s)-\sum_{k=0}^{n-1}s^{n-1-k}f^{(k)}(0)$
$\int_0^t f(\tau)d\tau$	$\frac{1}{s}F(s)$
$tf(t)$	$-F'(s)$
$\frac{1}{t}f(t)$	$\int_s^\infty F(\sigma)d\sigma$
$f(t)*g(t)$	$F(s)G(s)$
$f(0)$	$\lim_{s\to\infty} sF(s)$
$f(\infty)$	$\lim_{s\to 0} sF(s)$

$f(t)$	$F(s)$	$f(t)$	$F(s)$
t	$\frac{1}{s^2}$	$\delta(t)$	1
t^2	$\frac{2}{s^3}$	1 or $u(t)$	$\frac{1}{s}$
$t^n,\ n=0,1,2,\ldots$	$\frac{n!}{s^{n+1}}$	$u(t-a)$	$\frac{1}{s}e^{-as}$
e^{at}	$\frac{1}{s-a}$	$\delta(t-a)$	e^{-as}
$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$	te^{-at}	$\frac{1}{(s+a)^2}$
$\cos\omega t$	$\frac{s}{s^2+\omega^2}$	$t\sin\omega t$	$\frac{2\omega s}{(s^2+\omega^2)^2}$
$\sinh\omega t$	$\frac{\omega}{s^2-\omega^2}$	$e^{at}\sin\omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$
$\cosh\omega t$	$\frac{s}{s^2-\omega^2}$	$e^{at}\cos\omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
$f'(t)$	$sF(s)-f(0)$		
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$		
$tf'(t)$	$-F(s)-sF'(s)$		
$tf''(t)$	$-2sF(s)-s^2F'(s)-f(0)$		

- Convolution: $f(t)*g(t)=\int_0^t f(\tau)g(t-\tau)d\tau$

5 Fourier Series and Fourier Transform in a Nutshell

5.1 Fourier Series

Fourier representation with period 2π	Fourier representation with period $p=2L$
$f(x) = a_0+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx)$	$f(x) = a_0+\sum_{k=1}^{\infty}\left(a_k\cos\frac{k\pi}{L}x+b_k\sin\frac{k\pi}{L}x\right)$
where: $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx$	where: $a_0 = \frac{1}{2L}\int_{-L}^{L}f(x)dx$
$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos kxdx$	$a_k = \frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{k\pi}{L}xdx$
$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin kxdx$	$b_k = \frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{k\pi}{L}xdx$

- Even function means $f(-x)=f(x)$; odd function means $f(-x)=-f(x)$. Examples are cosine and sine.

Even function	Odd function
$f(-x)=f(x)$	$f(-x)=-f(x)$
Example: $\cos x$	Example: $\sin x$
$\int_{-L}^{L}f_{\text{even}}(x)dx=2\int_0^L f_{\text{even}}(x)dx$	$\int_{-L}^{L}f_{\text{odd}}(x)dx=0$
$f_{\text{even}}(x)=a_0+\sum_{k=1}^{\infty}\left(a_k\cos\frac{k\pi}{K}x\right)$	$f_{\text{odd}}(x)=\sum_{k=1}^{\infty}\left(b_k\sin\frac{k\pi}{K}x\right)$

- Further properties of Fourier series representation:
  - $f(x) = f_1(x) + f_2(x)$ then the Fourier series is the sum of every corresponding coefficients
  - $cf(x)$ has the Fourier series with each Fourier coefficients of $f(x)$ multiplied by c
- Exponential representation of complex number: $e^{i\theta} = \cos\theta + i\sin\theta$
  - $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \cosh(i\theta)$
  - $\sin\theta = \frac{1}{2}(e^{i\theta} - e^{-i\theta}) = \sinh(i\theta)$
- Complex Fourier series:

Fourier representation with period 2π	Fourier representation with period $p = 2L$
$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$	$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$
where: $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$	where: $c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx$
and: $c_0 = a_0$	and: $c_0 = a_0$
$c_n = \frac{1}{2}(a_n - ib_n) \quad (n > 0)$	$c_n = \frac{1}{2}(a_n - ib_n) \quad (n > 0)$
$c_{-n} = \frac{1}{2}(a_n + ib_n) \quad (n > 0)$	$c_{-n} = \frac{1}{2}(a_n + ib_n) \quad (n > 0)$

- The integral for finding Fourier coefficients a_n, b_n, c_n can integrate for any complete period p, e.g. $a_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx$
  - Hence we usually writes: $c_n = \frac{1}{2L}\int_p f(x)e^{-2in\pi x/p}dx$ for the series $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2in\pi x/p}$

5.2 Approximation by trigonometric polynomials

- Parseval's relation: Given $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$, we have $\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx = 2a_0^2 + \sum_{n+1}^{\infty}(a_n^2 + b_n^2)$
- Approximation of periodic function by Fourier series up to $n = N$: $f(x) \approx F(x) = a_0 + \sum_{n=1}^{N}(a_n\cos nx + b_n\sin nx)$
  - The "square error": $E = \int_{-\pi}^{\pi}(f - F)^2dx = \int_{-\pi}^{\pi} f^2(x)dx - \pi\left[2a_0^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2)\right]$
  - E is minimum if a_n and b_n are the Fourier coefficients
  - The square error $E \geq 0$ by definition, i.e. we have the Bessel inequality: $2a_0^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2) \leq \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx$

5.3 Fourier Transform

- Rationale: Assume it is periodic. For example, $f(x) = x$, repeats as if $-L < x < L$. Then take the limit on $L \to \infty$.
  - Any integrable function is a sum of (may be infinitely many) trigonometric functions
  - Fourier coefficients are the magnitude of the corresponding frequency

- Fourier transform is identical to Laplace transform, by replacing $s = i\omega$, i.e., assuming pure imaginary s.

$f(t)$	$F(s)$	$f(t)$	$F(s)$
$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$	$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$	$\delta(t)$	1
$af(t)+bg(t)$	$aF(\omega)+bG(\omega)$	$e^{i\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$e^{i\omega_0 t}f(t)$	$F(\omega-\omega_0)$	$u(t)$	$\pi\delta(\omega)+\frac{1}{i\omega}$
$f(t-t_0)$	$e^{-i\omega t_0}F(\omega)$	$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
$f(at)$	$\frac{1}{a}F(\frac{\omega}{a})$	$\sin\omega_0 t$	$-i\pi[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
$F(t)$	$2\pi f(-\omega)$	$u(t)\cos\omega_0 t$	$\frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\frac{i\omega}{\omega_0^2-\omega^2}$
$f^{(n)}(t)$	$(i\omega)^n F(\omega)$	$u(t)\sin\omega_0 t$	$\frac{-i\pi}{2}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\frac{\omega^2}{\omega_0^2-\omega^2}$
$(-it)^n f(t)$	$F^{(n)}(\omega)$	$u(t)e^{-at}\cos\omega_0 t$	$\frac{a+i\omega}{\omega_0^2+(a+i\omega)^2}$
$\int_{-\infty}^{t} f(\tau)d\tau$	$\frac{1}{i\omega}F(\omega)+\pi F(0)\delta(\omega)$	$u(t)e^{-at}\sin\omega_0 t$	$\frac{\omega_0}{\omega_0^2+(a+i\omega)^2}$
$f(t)*g(t)$	$\sqrt{2\pi}F(\omega)G(\omega)$	$u(t)e^{-at}$	$\frac{1}{a+i\omega}$
$f(t)g(t)$	$\frac{1}{2\pi}F(\omega)*G(\omega)$	$u(t)te^{-at}$	$\frac{1}{(a+i\omega)^2}$

A Important Stuff

$\cos(x\pm y)=\cos x\cos y\mp\sin x\sin y$	$\cos 2x=\cos^2 x-\sin^2 x$
$\sin(x\pm y)=\sin x\sin y\pm\cos x\cos y$	$\sin 2x = 2\sin x\cos x$
$2\cos^2 x = 1+\cos 2x$	$2\sin^2 x = 1-\cos 2x$
$2\cos x\cos y=\cos(x-y)+\cos(x+y)$	$\sin x+\sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$
$2\sin x\sin y=\cos(x-y)-\cos(x+y)$	$\sin x-\sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$
$2\sin x\cos y=\sin(x-y)+\sin(x+y)$	$\cos x+\cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$
$2\cos x\sin y=\sin(x+y)-\sin(x-y)$	$\cos x-\cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$

$\int\cos x dx=\sin x$	$\int e^{ax}dx=\frac{1}{a}e^{ax}$
$\int\sin x dx=-\cos x$	$\int xe^{ax}dx=e^{ax}\left(\frac{x}{a}-\frac{1}{a^2}\right)$
$\int\tan x dx=\ln\lvert\sec x\rvert$	$\int x^2e^{ax}dx=e^{ax}\left(\frac{x^2}{a}-\frac{2x}{a^2}-\frac{2}{a^3}\right)$
$\int\cot x dx=\ln\lvert\sin x\rvert$	$\int e^{ax}\sin bx dx=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)$
$\int\sec x dx=\ln\lvert\sec x+\tan x\rvert$	$\int e^{ax}\cos bx dx=\frac{e^{ax}}{a^2+b^2}(a\sin bx+b\cos bx)$
$\int x\cos x dx=\cos x+x\sin x$	$\int\frac{1}{a+bx}dx=\frac{1}{b}\ln\lvert a+bx\rvert$
$\int x\sin x dx=\sin x-x\cos x$	$\int\frac{1}{a^2+b^2x^2}dx=\frac{1}{ab}\tan^{-1}\left\lvert\frac{bx}{a}\right\rvert$
$\int x^2\cos x dx=2x\cos x+(x^2-2)\sin x$	$\int\frac{1}{\sqrt{a^2-b^2x^2}}dx=\frac{1}{b}\sin^{-1}\frac{bx}{a}$
$\int x^2\sin x dx=2x\sin x+(x^2-2)\cos x$	$\int\frac{1}{x\sqrt{b^2x^2-a^2}}dx=\frac{1}{a}\sec^{-1}\frac{bx}{a}$
$\int\ln x dx=x\ln x-x$	$\int\cosh x dx=\sinh x$
$\int\frac{1}{x\ln x}dx=\ln\lvert\ln x\rvert$	$\int\sinh x dx=-\cosh x$