Remedial Lesson 1: Review of Calculus Techniques

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1 Definition of Differentiation

- Differentiation as a limit:

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\frac{d}{dx}f(x)=f'(x)$$

  and __$f'(x)$__ is called the derivative of $f(x)$.

- If we think $f(x)$ is a ...
  1. ...is a curve, then $f'(x)$ is the __slope__ of the curve at the coordinate __$(x, f(x))$__
  2. ...is the displacement of a motion, and$x$ is time, then $f'(x)$ is the __velocity__
  3. ...is the velocity of a motion, and$x$ is time, then $f'(x)$ is the __acceleration__
  4. ...is the quantity of something (e.g. $$ in bank), and$x$ is time, then $f'(x)$ is the __rate of change__ of the quantity

- Sometimes, we may __approximate__ the derivative by

$$f'(x)=\frac{\Delta f}{\Delta x}$$

- Higher differenatials: If $y=f(x)$,

$$\begin{aligned}\frac{dy}{dx}&=\frac{d}{dx}f(x)=f'(x)\\ \frac{d^2y}{dx^2}&=\frac{d^2}{dx^2}f(x)=\frac{d}{dx}f'(x)\\ f^{(n)}(x)&=\underbrace{\frac{d}{dx}\frac{d}{dx}\cdots\frac{d}{dx}}_{n}f(x)\\ &=\frac{d^n}{dx^n}f(x)\\ &=\frac{d}{dx}f^{(n-1)}(x)\end{aligned}$$

2 Techniques of Differentiation

- Formulae to remember:

$$\begin{aligned}\frac{d}{dx}x^n&=nx^{n-1}\\ \frac{d}{dx}\frac{1}{x^n}&=\frac{d}{dx}x^{-n}=-nx^{-n-1}\\ \frac{d}{dx}\sin x&=\cos x\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}\cos x &= -\sin x \\
\frac{d}{dx}\tan x &= \sec^2 x \\
\frac{d}{dx}\cot x &= -\csc^2 x \\
\frac{d}{dx}\sec x &= \sec x \tan x \\
\frac{d}{dx}\csc x &= -\csc x \cot x
\end{aligned}$$

- Differentiation rules:

$$\begin{aligned}
\frac{d}{dx}k &= 0 \quad (k \text{ is constant}) \\
\frac{d}{dx}kf(x) &= k\frac{d}{dx}f(x) \quad k \text{ is constant}) \\
\frac{d}{dx}[f(x) \pm g(x)] &= \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x) \\
\frac{d}{dx}f(x)g(x) &= f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x) \\
\frac{d}{dx}\frac{f(x)}{g(x)} &= \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{g^2(x)} \\
\frac{d}{dx}f(g(x)) &= \frac{d}{dg}f(g)\frac{d}{dx}g(x)
\end{aligned}$$

Exercises

1. Evaluate $\frac{d}{dx}x^3 = \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h}$

$$\begin{aligned}
\frac{d}{dx}x^3 &= \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h\to 0}\frac{x^3 + 3hx^2 + 3h^2x + h^3 - x^3}{h} \\
&= \lim_{h\to 0}(3x^2 + 3hx + h^2) \\
&= 3x^2
\end{aligned}$$

2. Evaluate the derivative for $2x^2 + 13x + 15$

$$\begin{aligned}
\frac{d}{dx}(2x^2 + 13x + 15) &= \frac{d}{dx}(2x^2) + \frac{d}{dx}(13x) + \frac{d}{dx}(15) \\
&= 2\frac{d}{dx}(x^2) + 13\frac{d}{dx}(x) + 0 \\
&= 2(2x) + 13 \\
&= 4x + 13
\end{aligned}$$

3. Using the product rule and chain rule to find the derivative of $2x^2+13x+15=(2x+3)(x+5)$

$$\begin{aligned}\frac{d}{dx}[(2x+3)(x+5)] &= (2x+3)\frac{d}{dx}(x+5)+(x+5)\frac{d}{dx}(2x+3)\\ &= (2x+3)(1)+(x+5)(2)\\ &= 2x+3+2x+10\\ &= 4x+13\end{aligned}$$

4. Evaluate $\frac{d}{dx}(2x+3)^{99}$

$$\begin{aligned}\frac{d}{dx}(2x+3)^{99} &= 99(2x+3)^{98}\left(\frac{d}{dx}(2x+3)\right)\\ &= 99(2x+3)^{98}(2)\\ &= 198(2x+3)^{98}\end{aligned}$$

5. Evaluate the following:

$$\begin{aligned}\frac{d}{dx}\frac{x^2+2x+1}{x^2-2x+1} &= \frac{(x^2-2x+1)\frac{d}{dx}(x^2+2x+1)-(x^2+2x+1)\frac{d}{dx}(x^2-2x+1)}{(x^2-2x+1)^2}\\ &= \frac{(x^2-2x+1)(2x+2)-(x^2+2x+1)(2x-2)}{(x^2-2x+1)^2}\\ &= \frac{2(x-1)^2(x+1)-2(x+1)^2(x-1)}{(x-1)^4}\\ &= \frac{2(x-1)(x+1)[(x-1)-(x+1)]}{(x-1)^4}\\ &= \frac{-4(x-1)(x+1)}{(x-1)^4}\\ &= -\frac{4(x+1)}{(x-1)^3}\end{aligned}$$

6. Evaluate the following:

$$\begin{aligned}\frac{d}{dx}\frac{x^2+2x+1}{x^2-2x+1} &= \frac{d}{dx}\frac{(x+1)^2}{(x-1)^2}\\ &= \frac{(x-1)^2\frac{d}{dx}(x+1)^2-(x+1)^2\frac{d}{dx}(x-1)^2}{(x-1)^4}\\ &= \frac{2(x-1)^2(x+1)(1)-2(x+1)^2(x-1)(1)}{(x-1)^4}\\ &= \frac{2(x-1)(x+1)[(x-1)-(x+1)]}{(x-1)^4}\\ &= \frac{-4(x-1)(x+1)}{(x-1)^4}\\ &= -\frac{4(x+1)}{(x-1)^3}\end{aligned}$$

7. Evaluate the following:

$$\frac{d}{dx}\frac{x^2+2x+1}{x^2-2x+1} = \frac{d}{dx}\frac{(x+1)^2}{(x-1)^2} = \frac{d}{dx}\left[(x+1)^2(x-1)^{-2}\right]$$

$$
\begin{aligned}
&= 2(x+1)(x-1)^{-2} + (-2)(x-1)^{-3}(x+1)^2 \\
&= 2(x+1)(x-1)^{-2} - 2(x-1)^{-3}(x+1)^2 \\
&= 2(x+1)(x-1)^{-3}[(x-1)-(x+1)] \\
&= -4(x+1)(x-1)^{-3} \\
&= -\frac{4(x+1)}{(x-1)^3}
\end{aligned}
$$

8. Evaluate $\frac{d^2}{dx^2}\sqrt{x} = \frac{d}{dx}\left(\frac{d}{dx}\sqrt{x}\right)$

$$
\begin{aligned}
\frac{d^2}{dx^2} &= \frac{d}{dx}\left(\frac{d}{dx}\sqrt{x}\right) \\
&= \frac{d}{dx}\left(\frac{1}{2\sqrt{x}}\right) \\
&= \frac{d}{dx}\left(\frac{1}{2}x^{-1/2}\right) \\
&= \frac{-1}{4}x^{-3/2} \\
&= -\frac{1}{4\sqrt{x^3}}
\end{aligned}
$$

9. Evaluate $\frac{d}{dx}\left(3\sin^2(x^2)\right)$

$$
\begin{aligned}
\frac{d}{dx}\left(3\sin^2(x^2)\right) &= 6\sin(x^2)\frac{d}{dx}\left(\sin(x^2)\right) \\
&= 6\sin(x^2)\cos(x^2)\frac{d}{dx}x^2 \\
&= 12x\sin(x^2)\cos(x^2)
\end{aligned}
$$

3 Indefinite Integral

- Integration as the __inverse__ function of differentiation
- Examples:

$$
\begin{aligned}
\int nx^{n-1}dx &= x^n + C \\
\int \frac{-n}{x^{n+1}}dx &= \frac{1}{x^n} + C \\
\int \cos x dx &= \sin x + C \\
\int (-\sin x)dx &= \cos x + C \\
\int \sec^2 x dx &= \tan x + C \\
\int (-\csc^2 x)dx &= \cot x + C \\
\int \sec x \tan x dx &= \sec x + C \\
\int (-\csc x \cot x)dx &= \csc x + C
\end{aligned}
$$

- There is a *constant of integration* in the result of indefinite integral

4 Definite Integral

- Definition: Riemann Sum
  - Adding many many very very small quantities together

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{x_1=a}^{x_n=b} f(x_k)(x_{k+1}-x_k)$$

  - Example: Finding area under the curve $y = f(x)$, for $a \le x \le b$

- Fundamental Theorem of Calculus

$$\begin{aligned}\int_a^b f(x)dx &= \int f(x)dx\Big|_a^b \\ \int_a^b F'(x)dx &= F(b) - F(a)\end{aligned}$$

  - Example:

$$\begin{aligned}\int_0^1 x^2 dx &= \int x^2 dx\Big|_0^1 \\ &= \frac{1}{3}x^3\Big|_0^1 \\ &= \frac{1}{3}(1)^3 - \frac{1}{3}(0)^3 \\ &= \frac{1}{3}\end{aligned}$$

- Additional theorems to remember:

$$\begin{aligned}\int_a^b f(x)dx &= -\int_b^a f(x)dx \\ \int_a^a f(x)dx &= 0\end{aligned}$$

Exercises:

1. $\int_4^{121} \sqrt{x}dx$

$$\begin{aligned}\int_4^{121} \sqrt{x}dx &= \left[\frac{2}{3}x^{3/2}\right]_4^{121} \\ &= \left[\frac{2}{3}(121)^{3/2}\right] - \left[\frac{2}{3}4^{3/2}\right] \\ &= \frac{2}{3}[1331-8] \\ &= \frac{2}{3}(1323) \\ &= 882\end{aligned}$$

2. $\int_0^{4\pi} \cos x dx$

$$\begin{aligned}\int_0^{4\pi} \cos x dx &= [\sin x]_0^{4\pi} \\ &= [\sin(4\pi) - \sin 0]\end{aligned}$$

$$= 0$$

3. $\int_5^2 x^2 dx$

$$\begin{aligned}\int_5^2 x^2 dx &= \left[\frac{1}{3}x^3\right]_5^2 \\ &= \frac{1}{3}\left[2^3 - 5^3\right]_5^2 \\ &= \frac{1}{3}(8-125) \\ &= \frac{117}{3}\end{aligned}$$

4. $\int_0^1 mx^n dx$

$$\begin{aligned}\int_0^1 mx^n dx &= m\int_0^1 x^n dx \\ &= m\left[\frac{1}{n+1}x^{n+1}\right]_0^1 \\ &= \frac{m}{n+1}\left[1^{n+1} - 0^{n+1}\right] \\ &= \frac{m}{n+1}\end{aligned}$$

5. (Caution: Indefinite integral) $\int \sum_{k=1}^n B_k (kx+1)^k dx$

$$\begin{aligned}\int \sum_{k=1}^n B_k (kx+1)^k dx &= \sum_{k=1}^n \left(\int B_k (kx+1)^k dx\right) \\ &= \sum_{k=1}^n \left(B_k \int (kx+1)^k dx\right) \\ &= \sum_{k=1}^n \left(B_k \frac{1}{k(k+1)}(kx+1)^{k+1}\right) \\ &= \sum_{k=1}^n \frac{B_k (kx+1)^{k+1}}{k(k+1)}(kx+1)^{k+1}\end{aligned}$$

5 Bring-home Practices

5.1 Differentiation

1. Find $\dfrac{dy}{dx}$ for $y = \dfrac{\sin x}{x} + \dfrac{x}{\sin x}$
2. Find $\dfrac{dy}{dx}$ for $y = \dfrac{3x^5}{\sqrt[5]{x^2}} - \dfrac{7}{\sqrt[3]{x}} + 3\sqrt[7]{x^3}$
3. Find $\dfrac{dy}{dx}$ for $y = (x-1)(2x+1)(3-2x)$
4. Find $\dfrac{dy}{dx}$ for $y = \sqrt{\dfrac{1-x}{1+x}}$
5. Find $\dfrac{dy}{dx}$ for $y = \sin\left(\cos^2(x^3+x)\right)$

6. Find $\dfrac{dy}{dx}$ for $y = \dfrac{1}{x}\cos\dfrac{1}{x}$
7. Find $\dfrac{dy}{dx}$ for $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$
8. The slope of the tangent to the curve $y = ax^3 + bx$ at the point $(1,1)$ is -5. Calculate the values of a and b. With these values, find the coordinates of the points on the curve where the tangent is parallel to the x-axis.
9. (Caution: Implicit differentiation) Find the slopes at the points where $x = 2$ on the curve

$$17x^2 - 12xy + 8y^2 = 100$$

10. (Caution: Implicit differentiation) Find the slope of the tangent at the point $(\frac{7}{4}, 0)$ to the curve

$$2x^2y - 3y^2x - 4x + 5y + 7 = 0$$

11. Given $x^3 - 3axy + y^3 = b^3$, find $\dfrac{d^2y}{dx^2}$
12. Given $y = \dfrac{x^4}{144} + x^3 + 54x^2 + ax + b$ where a and b are constants. Find a if $\left(\dfrac{d^2y}{dx^2}\right)^2 = y$ for all values of x.
13. If $v = \dfrac{1}{r} + c$, show that $\dfrac{d^2v}{dr^2} + \dfrac{2}{r}\dfrac{dv}{dr} = 0$

5.2 Integration

1. Evaluate $\displaystyle\int \frac{1}{\cos^2 x} dx$
2. Evaluate $\displaystyle\int \frac{ax}{b} dx$
3. Evaluate $\displaystyle\int \frac{1}{\sqrt{2gh}} dh$
4. Evaluate $\displaystyle\int (x + 3\cos x) dx$
5. Evaluate $\displaystyle\int (3x-2)(4x+3) dx$
6. Evaluate $\displaystyle\int x^2(5-x)^4 dx$
7. Evaluate $\displaystyle\int (2x+3)^3 dx + \int (2x-3)^3 dx$
8. Evaluate $\displaystyle\int (\sqrt{x}+1)(x-\sqrt{x}+1) dx$
9. Evaluate $\displaystyle\int \frac{\sqrt[3]{x^2} - \sqrt[4]{x}}{\sqrt{x}} dx$
10. Evaluate $\displaystyle\int \frac{(1-x)^2}{\sqrt[3]{x^2}} dx$
11. Evaluate $\displaystyle\int \frac{(1-x)^3}{x\sqrt[3]{x}} dx$
12. A particle starts from a point O and moves in a straight line with a velocity v ms^{-1}, given by $v = 25t - 6t^2$, where t seconds is the time after leaving O. Calculate

    (a) the initial ($t = 0$) acceleration of the particle

(b) the value of t when the acceleration is zero

(c) the value of t when the particle returns to O (i.e. displacement is zero)