Remedial Lesson 2: All the Differentiation You Needed

Prepared by Adrian Sai-wah TAM (swtam3@ie.cuhk.edu.hk)

September 19, 2005

1 Table of Differentiations

Rules	Formula
Addition Rule	
Constant	
Product Rule	
Quotient Rule	
Chain Rule	
Parametric Function	$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$
Inverse Function	$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1}$

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
k		$\sin x$	
x		$\cos x$	
x^n		$\tan x$	
e^x		$\cot x$	
$\ln x$		$\sec x$	
		$\csc x$	

2 Exponential Functions

- We have something called ________________, $e = 2.717828...$

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

- Raising e to the power of x (i.e. $f(x) = e^x$) is called the ______________ function. For convenience, we may write it as $f(x) = \exp(x)$
- Differentiation:

$$\frac{d}{dx}e^x = e^x$$

- Example:

$$\frac{d}{dx}e^{-\lambda x} =$$
$$= e^{-\lambda x}\left(\qquad \right)$$
$$= -\lambda e^{-\lambda x}$$

3 Logarithmic Function

- If $y = e^x$, then we define $x = \ln y$. Where ln is the ___________________.
- Differentiation:

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

- Example:

$$\frac{d}{dx}\ln(x^2) = \frac{d}{d(x^2)}\ln(x^2)$$
$$=$$
$$= \frac{2}{x}$$

Exercises

1. Evaluate $\frac{d}{dx}\left[\ln(e^{2x}+e^{2a}) - \ln(e^{x-a}+e^{a-x}) + \frac{a}{x}\tan x\right]$

2. Evaluate $\frac{d}{dx}(\ln\ln x)$

$$\frac{d}{dx}(\ln\ln x)$$
$$=$$
$$=$$

3. Evaluate $\frac{d}{dx}\log_{10} x$

$$\begin{aligned}&\frac{d}{dx}\log_{10} x\\ =&\frac{d}{dx}\left(\qquad \right)\\ =&\\ =&\end{aligned}$$

4. Evaluate $\frac{d}{dx}\exp(\tan x^2)$

5. If $f(x) = e^{-x/a}\cos(\frac{x}{a})$, find $f(0) + af'(0)$

$$\begin{aligned}\frac{d}{dx}f(x) =&\frac{d}{dx}\left[e^{-x/a}\cos\left(\frac{x}{a}\right)\right]\\ =& \qquad\qquad\qquad + e^{-x/a}\frac{d}{dx}\cos\left(\frac{x}{a}\right)\\ =&\\ =&\\ =&-\frac{1}{a}e^{-x/a}\left[\cos\left(\frac{x}{a}\right)+\sin\left(\frac{x}{a}\right)\right]\\ \therefore \qquad f'(0) =&-\frac{1}{a}e^{0}\left[\cos(0)+\sin(0)\right]\\ =&\\ f(0) =&\\ \therefore \quad f(0)+af'(0) =&\end{aligned}$$

6. Show that $y = \exp(2x)\sin x$ satisfies $y'' - 4y' + 5y = 0$

4 Differentiation of Inverse Function

- Rule of Thumb:

$$\frac{dy}{dx} = \frac{1}{dx/dy}$$

4.1 Inverse Trigonometric Functions

- Example:

$$\begin{aligned}
\text{Let} \qquad x &= \sin y \\
\therefore \qquad y &= \sin^{-1} x \\
\frac{dx}{dy} &= \cos y \\
&= \sqrt{1 - \sin^2 y} \\
&= \\
\therefore \qquad \frac{dy}{dx} &= \\
\therefore \quad \frac{d}{dx}(\sin^{-1} x) &=
\end{aligned}$$

- Formulae to be used:

$$\begin{aligned}
\sin^2 x + \cos^2 x &= 1 \\
\sec^2 x &= \tan^2 x + 1 \\
\csc^2 x &= \cot^2 x + 1
\end{aligned}$$

- Complete the following table:

$f(x)$	$f'(x)$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	
$\tan^{-1} x$	
$\cot^{-1} x$	
$\sec^{-1} x$	
$\csc^{-1} x$	

4.2 Other inverse functions

- Example:

$$
\begin{aligned}
\text{Let} \qquad y &= \sqrt{x} \\
\therefore \qquad x &= y^2 \\
\frac{dx}{dy} &= 2y \\
&= 2\sqrt{x} \\
\frac{d}{dx}(\sqrt{x}) &= \frac{1}{2\sqrt{x}}
\end{aligned}
$$

Exercises

1. If $y = \sin^{-1} \frac{2x-7}{3}$, find $\frac{dy}{dx}$

2. If $y = \tan^{-1} e^x$, find $\frac{dy}{dx}$

3. If $y = \sec^{-1} \tan x$, find $\frac{dy}{dx}$

4. Show that if $y = (\sin^{-1} x)^2$, $(1 - x^2)y'' - xy' = 2$

5 Implicit Functions

- Implicit function: Those given as an equation, but not a function
- Example of ___________ function: Circle equation

$$(x-h)^2 + (y-k)^2 = r^2$$

- Example of ___________ function: Semicircle

$$y = k + \sqrt{r^2 - (x-h)^2}$$

- Differentiation of implicit function: Use the rules to differentiate both side, then simplify
- Example: Find $\frac{dy}{dx}$ from the standard form of circle equation

$$\begin{aligned} (x-h)^2 + (y-k)^2 &= r^2 \\ \therefore \quad 2(x-h) + 2(y-k)\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{x-h}{y-k} \end{aligned}$$

6 Application of Differentiation

6.1 Parametric functions

- Curves may be expressed as __________________, such as circle, it can be expressed in standard form:

$$x^2 + y^2 = r^2$$

  or in parametric form:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

- If a curve is presented in parametric form, the derivative $\frac{dy}{dx}$ is given by

$$\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta}$$

- Example: Use the parametric form of circle

$$\begin{cases} x = r\cos\theta + h \\ y = r\sin\theta + k \end{cases}$$

to find $\frac{dy}{dx}$.

$$\begin{aligned} x =& r\cos\theta + h \\ \therefore \quad \frac{dx}{d\theta} =& -r\sin\theta \\ y =& r\sin\theta + k \\ \therefore \quad \frac{dy}{d\theta} =& r\cos\theta \\ \therefore \quad \frac{dy}{dx} =& \\ =& \\ =& -\frac{x-h}{y-k} \end{aligned}$$

6.2 Find tangents and normals

- Tangents: Limit of chord on a curve
- Normals: Lines cutting the curve and perpendicular to the tangent at that point
- Differentation can help to find the slope, so that you can use straight line formulae to find the tangents or normals
- Example: Find the equation of tangent at point $(0,2)$ from the circle $x^2 + y^2 = 4$.

$$\begin{aligned} x^2 + y^2 =& 4 \\ \therefore \quad 2x + 2y\frac{dy}{dx} =& 0 \\ \frac{dy}{dx} =& \frac{x}{y} \\ \therefore \quad \left.\frac{dy}{dx}\right|_{(0,2)} =& \frac{0}{2} = 0 \\ \therefore \quad \text{Equation is: } y - 2 =& 0(x-0) \\ \implies \quad y =& 2 \end{aligned}$$

Exercises

1. Determine the constants A and B such that the normal to the curve $y = Ae^x + Be^{-x}$ at $(0,2)$ will be parallel to the line $3x - y = 4$

$$\begin{aligned}
y &= Ae^x + Be^{-x} \\
\frac{dy}{dx} &= Ae^x - Be^{-x} \\
\therefore \qquad \left.\frac{dy}{dx}\right|_{(0,2)} &= A - B = 3 \quad \text{(slope)} \\
2 &= Ae^0 + Be^0 = A + B \quad \text{(point)} \\
\implies \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} &= \begin{bmatrix} 3 \\ 2 \end{bmatrix} \\
\begin{bmatrix} A \\ B \end{bmatrix} &= \begin{bmatrix} \\ \\ \end{bmatrix}
\end{aligned}$$

2. Prove that curves $y = \exp(x^2)/ex$ and $y = x^2 - \ln x^3$ intersect at right angles at the point $(1,1)$

3. Find the equation of the tangent at $t = t_1$ to the curve given by the parametric equations $\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}$

4. Find the equations of tangent and normal at the point $(4,3)$ to the curve given by the parametric equations $x = t^2$ and $y = 2t - 1$. Show that the normal cuts the curve again at the point where $t = -3$.

5. Find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $t = \frac{\pi}{2}$ in the following parametric equations: $\begin{cases} x = a(nt - \sin t) \\ y = a(t + \sin t) \end{cases}$