Remedial Lesson 4: More Indefinite Integrals

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Formulas for Integration

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
1	x	$\sin kx$	$-\frac{1}{k}\cos kx$	$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}$
x^n	$\frac{x^{n+1}}{n+1}$	$\cos kx$	$\frac{1}{k}\sin kx$	$\frac{1}{\sqrt{x^2-a^2}}$	$\ln\left\lvert x+\sqrt{x^2-a^2}\right\rvert$
a^{kx}	$\frac{a^{kx}}{(k\ln a)}$	$\tan x$	$\ln\lvert\sec x\rvert$	$\frac{1}{\sqrt{x^2+a^2}}$	$\ln\left\lvert x+\sqrt{x^2+a^2}\right\rvert$
e^{kx}	$\frac{1}{k}e^{kx}$	$\cot x$	$\ln\lvert\sin x\rvert$	$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$
$\frac{1}{x}$	$\ln x$	$\sec x$	$\ln\lvert\sec x+\tan x\rvert$	$\frac{1}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$
		$\csc x$	$\ln\lvert\csc x-\cot x\rvert$	$\sqrt{a^2-x^2}$	$\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$
		$\sec kx\tan kx$	$\frac{1}{k}\sec kx$	$\sqrt{x^2-a^2}$	$\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln\left\lvert x+\sqrt{x^2-a^2}\right\rvert$
		$\csc kx\cot kx$	$-\frac{1}{k}\csc kx$	$\sqrt{x^2+a^2}$	$\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ln\left\lvert x+\sqrt{x^2+a^2}\right\rvert$
		$\sec^2 kx$	$\frac{1}{k}\tan kx$		
		$\csc^2 kx$	$-\frac{1}{k}\cot kx$		

1 Integration by Part

- This is the product rule:

$$\begin{aligned}
\frac{d}{dx}f(x)g(x) &= f(x)\frac{dg}{dx}+g(x)\frac{df}{dx}\\
\int\frac{d}{dx}f(x)g(x)dx &= \int f(x)\frac{dg}{dx}dx+\int g(x)\frac{df}{dx}dx\\
f(x)g(x) &= \int f(x)\frac{dg}{dx}dx+\int g(x)\frac{df}{dx}dx\\
\int f(x)g'(x)dx &= f(x)g(x)-\int g(x)f'(x)dx
\end{aligned}$$

which has the form:

$$\int uv'dx = uv-\int vu'dx$$

or we memorize this as:

$$\int udv = uv-\int vdu$$

- Example of use:

$$\begin{aligned}
&\int \ln x dx \\
=&x\ln x - \int x d(\ln x) \\
=&x\ln x - \int x \cdot \frac{1}{x} dx \\
=&x\ln x - \int (1) dx \\
=&x\ln x - x + C
\end{aligned}$$

Examples

1. Evaluate $\int x^3 \ln x dx$

$$\begin{aligned}
&\int x^3 \ln x dx \\
=&\int \ln x d(\frac{1}{4}x^4) \\
=&\frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^4 d(\ln x) \\
=&\frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3 dx \\
=&\frac{1}{4}x^4 \ln x - \frac{1}{4}\left(\frac{1}{4}x^4\right) + C \\
=&\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C
\end{aligned}$$

2. Evaluate $\int x^2 e^x dx$

$$\begin{aligned}
&\int x^2 e^x dx \\
=&\int x^2 d(e^x) \\
=&x^2 e^x - \int e^x d(x^2) \\
=&x^2 e^x - 2\int x e^x dx \\
=&x^2 e^x - 2\int x d(e^x) \\
=&x^2 e^x - 2\left[x e^x - \int e^x dx\right] \\
=&x^2 e^x - 2x e^x + 2e^x + C \\
=&e^x(x^2 - 2x + 2) + C
\end{aligned}$$

3. Evaluate $\int \frac{x\exp(x)}{(1+x)^2} dx$

$$\begin{aligned}
&\int \frac{xe^x}{(1+x)^2} dx \\
=&-\int xe^x d\left(\frac{1}{1+x}\right) \\
=&-\frac{xe^x}{1+x} + \int \frac{1}{1+x} d(xe^x) \\
=&-\frac{xe^x}{1+x} + \int \frac{e^x + xe^x}{1+x} dx
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{xe^x}{1+x} + \int \frac{e^x(1+x)}{1+x}dx \\
&= -\frac{xe^x}{1+x} + \int e^x dx \\
&= -\frac{xe^x}{1+x} + e^x + C
\end{aligned}
$$

Exercise

1. Evaluate $\int \sec^3 x dx$

$$
\begin{aligned}
\int \sec^3 x dx &= \int \sec x d(\tan x) \\
&= \sec x \tan x - \int \tan x d(\sec x) \\
&= \sec x \tan x - \int \tan^2 x \sec x dx \\
&= \sec x \tan x - \int (\sec^2 x - 1)\sec x dx \\
&= \sec x \tan x - \int (\sec^3 x - \sec x) dx \\
&= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx \\
&= \sec x \tan x - \int \sec^3 x dx + \ln|\sec x + \tan x| + C \\
\Longrightarrow \quad 2\int \sec^3 x dx &= \sec x \tan x + \ln|\sec x + \tan x| + C \\
\therefore \quad \int \sec^3 x dx &= \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C'
\end{aligned}
$$

2. Evaluate $\int \sin^{-1}\frac{x}{a} dx$

$$
\begin{aligned}
\int \sin^{-1}\frac{x}{a} dx &= x\sin^{-1}\frac{x}{a} - \int x d(\sin^{-1}\frac{x}{a}) \\
&= x\sin^{-1}\frac{x}{a} - \frac{1}{a}\int \frac{x}{\sqrt{1-x^2/a^2}}dx \\
&= x\sin^{-1}\frac{x}{a} - \frac{1}{a}\int \frac{1}{\sqrt{1-x^2/a^2}} \cdot \frac{d(1-x^2/a^2)}{-2/a^2} \\
&= x\sin^{-1}\frac{x}{a} - \frac{1}{a}\int \left(1-\frac{x^2}{a^2}\right)^{-1/2} \cdot \frac{d(1-x^2/a^2)}{-2/a^2} \\
&= x\sin^{-1}\frac{x}{a} - \frac{1}{a}\frac{2}{-2/a^2}\left(1-\frac{x^2}{a^2}\right)^{1/2} + C \\
&= x\sin^{-1}\frac{x}{a} + a\sqrt{1-\frac{x^2}{a^2}} + C \\
&= x\sin^{-1}\frac{x}{a} + \sqrt{a^2-x^2} + C
\end{aligned}
$$

3. Evaluate $\int x^2 \sin 2x dx$

$$\begin{aligned}
\int x^2 \sin 2x dx &= -\frac{1}{2}\int x^2 d(\cos 2x) \\
&= -\frac{1}{2}x^2 \cos 2x + \frac{1}{2}\int \cos 2x d(x^2) \\
&= -\frac{1}{2}x^2 \cos 2x + \frac{1}{2}\int \cos 2x(2x) dx \\
&= -\frac{1}{2}x^2 \cos 2x + \int x \frac{d(\sin 2x)}{2} \\
&= -\frac{1}{2}x^2 \cos 2x + \frac{1}{2}x \sin 2x - \frac{1}{2}\int \sin 2x dx \\
&= -\frac{1}{2}x^2 \cos 2x + \frac{1}{2}x \sin 2x - \frac{1}{4}\int \sin 2x d(2x) \\
&= -\frac{1}{2}x^2 \cos 2x + \frac{1}{2}x \sin 2x + \frac{1}{4}\cos 2x + C \\
&= \frac{1}{4}\left(1-2x^2\right)\cos 2x + \frac{1}{2}x \sin 2x + C
\end{aligned}$$

4. Evaluate $\int e^{2x} \cos^2 3x dx$

$$\begin{aligned}
\int e^{2x}\cos^2 3x dx &= \frac{1}{2}\int \cos^2 3x d(e^{2x}) \\
&= \frac{1}{2}e^{2x}\cos^2 3x - \frac{1}{2}\int e^{2x} d(\cos^2 3x) \\
&= \frac{1}{2}e^{2x}\cos^2 3x - \int e^{2x}\cos 3x(-\sin 3x)(3) dx \\
&= \frac{1}{2}e^{2x}\cos^2 3x + 3\int e^{2x}\cos 3x \sin 3x dx \\
&= \frac{1}{2}e^{2x}\cos^2 3x + \frac{3}{2}\int e^{2x}\cos 6x dx \\
&= \frac{1}{2}e^{2x}\cos^2 3x + \frac{3}{4}\int \cos 6x d(e^{2x}) \\
&= \frac{1}{2}e^{2x}\cos^2 3x + \frac{3}{4}e^{2x}\cos 6x - \frac{3}{4}\int e^{2x} d(\cos 6x) \\
&= \frac{1}{2}e^{2x}\cos^2 3x + \frac{3}{4}e^{2x}\cos 6x + \frac{9}{2}\int e^{2x}\sin 6x dx \\
&= \frac{1}{2}e^{2x}\cos^2 3x + \frac{3}{4}e^{2x}\cos 6x + \frac{9}{4}\int \sin 6x d(e^{2x}) \\
&= \frac{1}{2}e^{2x}\cos^2 3x + \frac{3}{4}e^{2x}\cos 6x + \frac{9}{4}e^{2x}\sin 6x - \frac{9}{4}\int e^{2x} d(\sin 6x) \\
&= \frac{1}{2}e^{2x}\cos^2 3x + \frac{3}{4}e^{2x}\cos 6x + \frac{9}{4}e^{2x}\sin 6x - \frac{27}{2}\int e^{2x}\cos 6x dx \\
\therefore \quad \int e^{2x}\cos^2 3x dx &= \frac{1}{2}e^{2x}\cos^2 3x + \frac{3}{2}\int e^{2x}\cos 6x dx \\
&= \frac{1}{2}e^{2x}\cos^2 3x + \frac{3}{4}e^{2x}\cos 6x + \frac{9}{4}e^{2x}\sin 6x - \frac{27}{2}\int e^{2x}\cos 6x dx \\
\frac{3}{2}\int e^{2x}\cos 6x dx &= \frac{3}{4}e^{2x}\cos 6x + \frac{9}{4}e^{2x}\sin 6x - \frac{27}{2}\int e^{2x}\cos 6x dx \\
15\int e^{2x}\cos 6x dx &= \frac{3}{4}e^{2x}\cos 6x + \frac{9}{4}e^{2x}\sin 6x \\
\int e^{2x}\cos 6x dx &= \frac{1}{20}e^{2x}\cos 6x + \frac{3}{20}e^{2x}\sin 6x \\
\therefore \quad \int e^{2x}\cos^2 3x dx &= \frac{1}{2}e^{2x}\cos^2 3x + \frac{3}{2}\left(\frac{1}{20}e^{2x}\cos 6x + \frac{3}{20}e^{2x}\sin 6x\right) \\
&= \frac{1}{2}e^{2x}\cos^2 3x + \frac{3}{40}e^{2x}\cos 6x + \frac{9}{40}e^{2x}\sin 6x
\end{aligned}$$

5. Evaluate $\int \sqrt{x^2-a^2}dx$

$$\begin{aligned}
\int \sqrt{x^2-a^2}dx &= \int a\sec t\tan t\sqrt{a^2\sec^2 t-a^2}dt && \text{(sub } x = a\sec t\text{)}\\
&= a^2\int \sec t\tan^2 t dt\\
&= a^2\int \tan t d(\sec t)\\
&= a^2\sec t\tan t - a^2\int \sec t d(\tan t)\\
&= a^2\sec t\tan t - a^2\int \sec^3 t dt\\
&= a^2\sec t\tan t - \frac{a^2}{2}\sec t\tan t - \frac{a^2}{2}\ln|\sec t+\tan t| + C' && \text{(use result of \#1)}\\
&= \frac{a^2}{2}\sec t\tan t - \frac{a^2}{2}\ln|\sec t+\tan t| + C\\
&= \frac{1}{2}x\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|\frac{x}{a}+\sqrt{\frac{x^2}{a^2}-1}\right| + C\\
&= \frac{1}{2}x\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + C'
\end{aligned}$$

6. Evaluate $\int \sqrt{a^2+x^2}dx$

$$\begin{aligned}
\int \sqrt{a^2+x^2}dx &= \int a\sec^2 t\sqrt{a^2\tan^2 t+a^2}dt && \text{(sub } x = a\tan t\text{)}\\
&= a^2\int \sec^3 t dt\\
&= \frac{a^2}{2}\sec t\tan t + \frac{a^2}{2}\ln|\sec t+\tan t| + C && \text{(use result of \#1)}\\
&= \frac{1}{2}x\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|\sqrt{\frac{x^2}{a^2}+1}+\frac{x}{a}\right| + C\\
&= \frac{1}{2}x\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|\sqrt{x^2+a^2}+x\right| + C'
\end{aligned}$$

2 Partial Fractions, and Integration of Rational Funtions

2.1 Partial fractions

- Partial fractions:

$$\frac{1}{3x^2-2x-1} = \frac{1/4}{x-1} + \frac{-3/4}{3x+1}$$

- In general, a fraction whose numerator and denominator are both polynomial with real coefficients can be expressed as a series of similar fractions, the fractional terms of the series is either degree 0, 1, or 2

- How to do? As follows:

$$
\begin{aligned}
\frac{1}{3x^2-2x-1} &= \frac{1}{(x-1)(3x+1)} \\
&\equiv \frac{A}{x-1}+\frac{B}{3x+1} && \text{(for some unknowns } A, B\text{)} \\
1 &\equiv A(3x+1)+B(x-1) && \text{(multiply each side by } 3x^2-2x-1\text{)} \\
1 &= A-B && \text{(when } x=0\text{)} \\
1 &= (3A+B)x+(A-B) \\
\therefore\quad 3A+B &= 0 && \text{(properties of identity)} \\
3A &= -B \\
A &= \frac{1}{4} && \text{(as } A-B=1\text{)} \\
B &= -\frac{3}{4}
\end{aligned}
$$

1. Factorize the denominator of the polynomial fraction into products of degree 1 of 2 polynomials
2. Each factor of the denominator become the denominator of a separate a fraction. If there are factors raised to higher powers, each power is a denominator
3. Numerators are unknown polynomials of a lower degree, to be solved by various method
4. Sum of them should be identical to the original polynomial fraction

   – One of the best way to solve for (numerators of) partial fractions is *the method of undetermined coefficients*

Exercises

1. Express as partial fractions for $\dfrac{x^2+x+1}{x^3-4x^2+x+6}$

$$
\begin{aligned}
\frac{x^2+x+1}{x^3-4x^2+x+6} &= \frac{x^2+x+1}{(x+1)(x-2)(x-3)} && \text{(factorize)} \\
&\equiv \frac{A}{x+1}+\frac{B}{x-2}+\frac{C}{x-3} \\
x^2+x+1 &= A(x-2)(x-3)+B(x+1)(x-3)+C(x+1)(x-2) \\
1 &= A(-3)(-4) && \text{(sub } x=-1\text{)} \\
\therefore\quad A &= 1/12 \\
7 &= B(3)(-1) && \text{(sub } x=2\text{)} \\
\therefore\quad B &= -7/3 \\
13 &= C(4)(1) && \text{(sub } x=3\text{)} \\
\therefore\quad C &= 13/4 \\
\therefore\quad \frac{x^2+x+1}{x^3-4x^2+x+6} &= \frac{1/12}{x+1}+\frac{-7/3}{x-2}+\frac{13/4}{x-3} \\
&= \frac{1}{12(x+1)}-\frac{7}{3(x-2)}+\frac{13}{4(x-3)}
\end{aligned}
$$

2. Express as partial fractions for $\dfrac{x^2}{(x+1)(x-1)^3}$

$$
\begin{aligned}
\frac{x^2}{(x+1)(x-1)^3} &\equiv \frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{(x-1)^2}+\frac{D}{(x-1)^3} \\
x^2 &= A(x-1)^3+B(x+1)(x-1)^2+C(x+1)(x-1)+D(x+1) \\
1 &= 2D && \text{(sub } x=1\text{)} \\
\therefore\quad D &= 1/2 \\
1 &= -8A && \text{(sub } x=-1\text{)} \\
\therefore\quad A &= -1/8 \\
2x &= 3A(x-1)^2+B[(x-1)^2+2(x+1)(x-1)]+2Cx+D && \text{(differentiate)} \\
2 &= 2C+\frac{1}{2} && \text{(sub } x=1\text{)} \\
\therefore\quad C &= \frac{3}{4} \\
0 &= \frac{-3}{8}+B[1-2]+\frac{1}{2} && \text{(sub } x=0\text{)} \\
\therefore\quad B &= \frac{1}{8} \\
\therefore\quad \frac{x^2}{(x+1)(x-1)^3} &= -\frac{1}{8(x+1)}+\frac{1}{8(x-1)}+\frac{3}{4(x-1)^2}+\frac{1}{2(x-1)^3}
\end{aligned}
$$

3. Express as partial fractions for $\dfrac{3x^3-2x-20}{(x^2+3)(2x^2-6x+5)}$

$$
\begin{aligned}
\frac{3x^3-2x-20}{(x^2+3)(2x^2-6x+5)} &\equiv \frac{Ax+B}{x^2+3}+\frac{Cx+D}{2x^2-6x+5} && \text{(no further factors)} \\
3x^3-2x-20 &= (Ax+B)(2x^2-6x+5)+(Cx+D)(x^2+3) \\
-20 &= 5B+3D && \text{(sub } x=0\text{)} \\
3 &= 2A+C && \text{(coeff. } x^3\text{)} \\
0 &= 2B-6A+D \\
-2 &= 5A-6B+3C
\end{aligned}
$$

$$
\therefore\quad \begin{bmatrix} 0 & 5 & 0 & 3 \\ 2 & 0 & 1 & 0 \\ -6 & 2 & 0 & 1 \\ 5 & -6 & 3 & 0 \end{bmatrix}\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} -20 \\ 3 \\ 0 \\ -2 \end{bmatrix}
$$

$$
\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 5 \\ -10 \end{bmatrix}
$$

$$
\therefore\quad \frac{3x^3-2x-10}{(x^2+3)(2x^2-6x+5)} = \frac{-x+2}{x^2+3}+\frac{5x-10}{2x^2-6x+5}
$$

2.2 Integration using Partial Fractions

- Integration of $\int \frac{1}{Ax+B}dx$ can be solved by substituting $y = Ax+B$

$$\begin{aligned}\int \frac{dx}{Ax+B} &= \frac{1}{A}\int \frac{d(Ax+B)}{Ax+B} \\ &= \frac{1}{A}\ln|Ax+B| + C\end{aligned}$$

- Integration of "constant over quadratic": Competing square!
  - Example:

$$\begin{aligned}\int \frac{1}{x^2+2x-3}dx &= \int \frac{1}{(x+1)^2-2^2}dx \\ &= \int \frac{2\sec t\tan t dt}{2(\sec^2 t-1)} && \text{(sub } x+1 = 2\sec t\text{)} \\ &= \int \frac{\sec t}{\tan t}dt \\ &= \int \csc t dt \\ &= \ln|\csc t - \cot t| + C \\ &= \ln\left|\frac{x+1}{\sqrt{(x+1)^2-2}} - \frac{2}{\sqrt{(x+1)^2-2}}\right| + C \\ &= \ln\left|\frac{x-1}{\sqrt{x^2+2x-3}}\right| + C\end{aligned}$$

- Integration of "linear over quadratic": Break into two!
  - Example:

$$\begin{aligned}\int \frac{4x+5}{x^2+2x-3}dx &= \int \frac{2(2x+2)+1}{x^2+2x-3}dx \\ &= 2\int \frac{2x+2}{x^2+2x-3}dx + \int \frac{1}{x^2+2x-3}dx \\ &= 2\int \frac{d(x^2+2x-3)}{x^2+2x-3} + \int \frac{1}{x^2+2x-3}dx \\ &= 2\ln\left|x^2+2x-3\right| + \ln\left|\frac{x-1}{\sqrt{x^2+2x-3}}\right| + C\end{aligned}$$

- Other kinds of fractions: Partial fractions!

- Example:

$$
\begin{aligned}
\int \frac{x^5+x^3-1}{x^2(x^2+1)^2}dx &= \int \left(\frac{x^5}{x^2(x^2+1)^2}+\frac{x^3}{x^2(x^2+1)^2}-\frac{1}{x^2(x^2+1)^2}\right)dx \\
&= \int \frac{x^3}{(x^2+1)^2}dx+\int \frac{x}{(x^2+1)^2}dx-\int \frac{1}{x^2(x^2+1)^2}dx \\
\int \frac{x^3}{(x^2+1)^2}dx &= \int \frac{x}{x^2+1}dx-\int \frac{x}{(x^2+1)^2}dx \\
&= \frac{1}{2}\ln\left|x^2+1\right|+\frac{1}{2}\cdot\frac{1}{x^2+1}+C_1 \\
\int \frac{x}{(x^2+1)^2}dx &= -\frac{1}{2}\cdot\frac{1}{x^2+1}+C_2 && \text{(see above!)} \\
\int \frac{dx}{x^2(x^2+1)^2} &= \int \frac{1}{x^2}dx+\int \frac{-1}{x^2+1}dx+\int \frac{-1}{(x^2+1)^2}dx \\
&= -\frac{1}{x}-\int \frac{1}{x^2+1}dx-\int \frac{(x^2+1)-x^2}{(x^2+1)^2}dx \\
&= -\frac{1}{x}-\int \frac{1}{x^2+1}dx-\int \frac{1}{x^2+1}dx+\int \frac{x^2}{(x^2+1)^2}dx \\
&= -\frac{1}{x}-2\int \frac{1}{x^2+1}dx+\frac{-1}{2}\int xd\left(\frac{1}{x^2+1}\right) \\
&= -\frac{1}{x}-2\tan^{-1}x-\frac{1}{2}\frac{x}{x^2+1}+\frac{1}{2}\int \frac{dx}{x^2+1} \\
&= -\frac{1}{x}-2\tan^{-1}x-\frac{1}{2}\frac{x}{x^2+1}+\frac{1}{2}\tan^{-1}x+C_3 \\
&= -\frac{1}{x}-\frac{3}{2}\tan^{-1}x-\frac{1}{2}\frac{x}{x^2+1}+C_3 \\
\therefore \quad \int \frac{x^5+x^3-1}{x^2(x^2+1)^2}dx &= \frac{1}{2}\ln\left|x^2+1\right|+\frac{1}{2}\cdot\frac{1}{x^2+1}+C_1-\frac{1}{2}\cdot\frac{1}{x^2+1}+C_2+\frac{1}{x}+\frac{3}{2}\tan^{-1}x+\frac{1}{2}\frac{x}{x^2+1}-C_3 \\
&= \frac{1}{2}\ln(x^2+1)+\frac{3}{2}\tan^{-1}x+\frac{3x^2+2}{2x(x^2+1)}+C
\end{aligned}
$$

Exercises

1. Evaluate $\int \frac{x^2+x+1}{x^3-4x^2+x+6}dx$

$$
\begin{aligned}
\int \frac{x^2+x+1}{x^3-4x^2+x+6}dx &= \int \left(\frac{1}{12(x+1)}-\frac{7}{3(x-2)}+\frac{13}{4(x-3)}\right)dx \\
&= \frac{1}{12}\int \frac{dx}{x+1}-\frac{7}{3}\int \frac{dx}{x-2}+\frac{13}{4}\int \frac{dx}{x-3} \\
&= \frac{1}{12}\ln|x+1|-\frac{7}{3}\ln|x-2|+\frac{13}{4}\ln|x-3|+C \\
&= \frac{1}{12}\ln\left|\frac{(x+1)(x-3)^{39}}{(x-2)^{28}}\right|+C
\end{aligned}
$$

2. Evaluate $\int \frac{x^2}{(x+1)(x-1)^3}dx$

$$\begin{aligned}\int \frac{x^2}{(x+1)(x-1)^3}dx &= \int \left(-\frac{1}{8(x+1)} + \frac{1}{8(x-1)} + \frac{3}{4(x-1)^2} + \frac{1}{2(x-1)^3}\right)dx \\ &= -\frac{1}{8}\int \frac{dx}{x+1} + \frac{1}{8}\int \frac{dx}{x-1} + \frac{3}{4}\int \frac{dx}{(x-1)^2} + \frac{1}{2}\int \frac{dx}{(x-1)^3} \\ &= -\frac{1}{8}\ln|x+1| + \frac{1}{8}\ln|x-1| - \frac{3}{4}\cdot\frac{1}{x-1} - \frac{1}{4}\cdot\frac{1}{(x-1)^2} + C \\ &= \frac{1}{8}\ln\left|\frac{x-1}{x+1}\right| + \frac{2-3x}{4(x-1)^2} + C\end{aligned}$$

3. Evaluate $\int \frac{3x^3-2x-20}{(x^2+3)(2x^2-6x+5)}dx$

$$\begin{aligned}\int \frac{3x^3-2x-20}{(x^2+3)(2x^2-6x+5)}dx &= \int \left(\frac{-x+2}{x^2+3} + \frac{5x-10}{2x^2-6x+5}\right)dx \\ &= -\int \frac{x}{x^2+3}dx + \int \frac{2}{x^2+3}dx + \int \frac{5x-10}{2x^2-6x+5}dx \\ &= -\frac{1}{2}\int \frac{2x}{x^2+3}dx + 2\int \frac{1}{x^2+(\sqrt{3})^2}dx + \frac{5}{4}\int \frac{4x-6}{2x^2-6x+5}dx - \frac{5}{4}\int \frac{2}{2x^2-6x+5}dx \\ &= -\frac{1}{2}\ln\left|x^2+3\right| + 2\left(\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\right) + \frac{5}{4}\ln\left|2x^2-6x+5\right| - \frac{5}{4}\int \frac{dx}{(x-\frac{3}{2})^2+(\frac{1}{2})^2} \\ &= -\frac{1}{2}\ln\left|x^2+3\right| + 2\left(\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\right) + \frac{5}{4}\ln\left|2x^2-6x+5\right| - \frac{5}{4}\left(\frac{1}{1/2}\tan^{-1}\frac{x-3/2}{1/2}\right) + C \\ &= -\frac{1}{2}\ln\left|x^2+3\right| + 2\left(\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\right) + \frac{5}{4}\ln\left|2x^2-6x+5\right| - \frac{5}{4}\left(2\tan^{-1}(2x-3)\right) + C \\ &= \frac{1}{4}\ln\left|\frac{(2x^2-6x+5)^5}{(x^2+3)^2}\right| + \frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - \frac{5}{2}\tan^{-1}(2x-3) + C\end{aligned}$$

3 Bring-home Practices

1. $\int \frac{(3x-1)}{x^2+9}dx$
2. $\int \frac{x}{\sqrt{27+6x-x^2}}dx$
3. $\int \frac{x}{(x+1)(x+3)(x+5)}dx$
4. $\int \frac{x^5-x^3+1}{x^4-x^3}dx$
5. $\int \sec^5 x dx$
6. $\int \tan^5 x dx$
7. $\int \frac{dx}{x\sqrt{x^2+3}}$
8. $\int \frac{xdx}{\sqrt{3+2x-x^2}}$
9. $\int e^{ax}\sin bx dx$