Remedial Lesson 4: More Indefinite Integrals

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September 29, 2005

Formulas for Integration

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
1	x	$\sin kx$	$-\frac{1}{k}\cos kx$	$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}$
x^n	$\frac{x^{n+1}}{n+1}$	$\cos kx$	$\frac{1}{k}\sin kx$	$\frac{1}{\sqrt{x^2-a^2}}$	$\ln\left\lvert x+\sqrt{x^2-a^2}\right\rvert$
a^{kx}	$\frac{a^{kx}}{(k\ln a)}$	$\tan x$	$\ln\lvert\sec x\rvert$	$\frac{1}{\sqrt{x^2+a^2}}$	$\ln\left\lvert x+\sqrt{x^2+a^2}\right\rvert$
e^{kx}	$\frac{1}{k}e^{kx}$	$\cot x$	$\ln\lvert\sin x\rvert$	$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$
$\frac{1}{x}$	$\ln x$	$\sec x$	$\ln\lvert\sec x+\tan x\rvert$	$\frac{1}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$
		$\csc x$	$\ln\lvert\csc x-\cot x\rvert$	$\sqrt{a^2-x^2}$	$\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$
		$\sec kx\tan kx$	$\frac{1}{k}\sec kx$	$\sqrt{x^2-a^2}$	$\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln\left\lvert x+\sqrt{x^2-a^2}\right\rvert$
		$\csc kx\cot kx$	$-\frac{1}{k}\csc kx$	$\sqrt{x^2+a^2}$	$\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ln\left\lvert x+\sqrt{x^2+a^2}\right\rvert$
		$\sec^2 kx$	$\frac{1}{k}\tan kx$		
		$\csc^2 kx$	$-\frac{1}{k}\cot kx$		

1 Integration by Part

- This is the product rule:

$$
\begin{aligned}
\frac{d}{dx}f(x)g(x) &= f(x)\frac{dg}{dx}+g(x)\frac{df}{dx} \\
\int\frac{d}{dx}f(x)g(x)dx &= \int f(x)\frac{dg}{dx}dx+\int g(x)\frac{df}{dx}dx \\
f(x)g(x) &= \int f(x)\frac{dg}{dx}dx+\int g(x)\frac{df}{dx}dx \\
\int f(x)g'(x)dx &= f(x)g(x)-\int g(x)f'(x)dx
\end{aligned}
$$

which has the form:

$$\int uv'dx = uv-\int vu'dx$$

or we memorize this as:

$$\int udv = uv-\int vdu$$

- Example of use:

$$\begin{aligned}
&\int \ln x dx \\
=&x\ln x - \int x d(\ln x) \\
=&x\ln x - \int x \cdot \frac{1}{x} dx \\
=&x\ln x - \int (1) dx \\
=&x\ln x - x + C
\end{aligned}$$

Examples

1. Evaluate $\int x^3 \ln x dx$

$$\begin{aligned}
&\int x^3 \ln x dx \\
=&\int \ln x d(\frac{1}{4}x^4) \\
=&\frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^4 d(\ln x) \\
=&\frac{1}{4}x^4 \ln x - \\
= & \\
= &
\end{aligned}$$

2. Evaluate $\int x^2 e^x dx$

$$\begin{aligned}
&\int x^2 e^x dx \\
=&\int x^2 d(e^x) \\
= & \\
=&x^2e^x - 2\int xe^x dx \\
=&x^2e^x - 2\int x d(e^x) \\
=&x^2e^x - \\
= & \\
= &
\end{aligned}$$

3. Evaluate $\int \frac{x\exp(x)}{(1+x)^2} dx$

$$\begin{aligned}
&\int \frac{xe^x}{(1+x)^2} dx \\
=&-\int xe^x d\left(\frac{1}{1+x}\right) \\
= & \\
=&-\frac{xe^x}{1+x} +
\end{aligned}$$

$$= -\frac{xe^x}{1+x} + \int \frac{e^x(1+x)}{1+x}dx$$

$$=$$

$$=$$

Exercise

1. Evaluate $\int \sec^3 x dx$

2. Evaluate $\int \sin^{-1} \frac{x}{a} dx$

3. Evaluate $\int x^2 \sin 2x dx$

4. Evaluate $\int e^{2x} \cos^2 3x dx$

5. Evaluate $\int \sqrt{x^2 - a^2} dx$

6. Evaluate $\int \sqrt{a^2 + x^2} dx$

2 Partial Fractions, and Integration of Rational Funtions

2.1 Partial fractions

- Partial fractions:

$$\frac{1}{3x^2 - 2x - 1} = \frac{1/4}{x - 1} + \frac{-3/4}{3x + 1}$$

- In general, a fraction whose numerator and denominator are both polynomial with real coefficients can be expressed as a series of similar fractions, the fractional terms of the series is either degree 0, 1, or 2

- How to do? As follows:

$$
\begin{aligned}
\frac{1}{3x^2-2x-1} &= \frac{1}{(x-1)(3x+1)} \\
&\equiv \frac{A}{x-1}+\frac{B}{3x+1} && \text{(for some unknowns } A, B\text{)} \\
1 &\equiv A(3x+1)+B(x-1) && \text{(multiply each side by } 3x^2-2x-1\text{)} \\
1 &= A-B && \text{(when } x=0\text{)} \\
1 &= (3A+B)x+(A-B) \\
\therefore \quad 3A+B &= 0 && \text{(properties of identity)} \\
3A &= -B \\
A &= \frac{1}{4} && \text{(as } A-B=1\text{)} \\
B &= -\frac{3}{4}
\end{aligned}
$$

1. Factorize the denominator of the polynomial fraction into products of degree 1 of 2 polynomials
2. Each factor of the denominator become the denominator of a separate a fraction. If there are factors raised to higher powers, each power is a denominator
3. Numerators are unknown polynomials of a lower degree, to be solved by various method
4. Sum of them should be identical to the original polynomial fraction

– One of the best way to solve for (numerators of) partial fractions is *the method of undetermined coefficients*

Exercises

1. Express as partial fractions for $\frac{x^2+x+1}{x^3-4x^2+x+6}$

$$
\begin{aligned}
\frac{x^2+x+1}{x^3-4x^2+x+6} &= \frac{x^2+x+1}{(x+1)(x-2)(x-3)} && \text{(factorize)} \\
&\equiv \frac{A}{x+1}+\frac{B}{x-2}+\frac{C}{x-3} \\
x^2+x+1 &= \\
1 &= A(-3)(-4) && \text{(sub } x=-1\text{)} \\
\therefore \quad A &= \\
7 &= B(3)(-1) && \text{(sub } x=2\text{)} \\
\therefore \quad B &= \\
13 &= C(4)(1) && \text{(sub } x=3\text{)} \\
\therefore \quad C &= \\
\therefore \quad \frac{x^2+x+1}{x^3-4x^2+x+6} &= \frac{\quad}{x+1}+\frac{\quad}{x-2}+\frac{\quad}{x-3} \\
&=
\end{aligned}
$$

2. Express as partial fractions for $\dfrac{x^2}{(x+1)(x-1)^3}$

3. Express as partial fractions for $\dfrac{3x^3-2x-20}{(x^2+3)(2x^2-6x+5)}$

2.2 Integration using Partial Fractions

- Integration of $\int \frac{1}{Ax+B}dx$ can be solved by substituting $y = Ax+B$

$$\begin{aligned}\int \frac{dx}{Ax+B} &= \frac{1}{A}\int \frac{d(Ax+B)}{Ax+B} \\ &= \frac{1}{A}\ln|Ax+B|+C\end{aligned}$$

- Integration of "constant over quadratic": Competing square!
  - Example:

$$\begin{aligned}\int \frac{1}{x^2+2x-3}dx &= \int \frac{1}{(x+1)^2-2^2}dx \\ &= \int \frac{2\sec t\tan t dt}{2(\sec^2 t-1)} && \text{(sub } x+1=2\sec t\text{)} \\ &= \int \qquad dt \\ &= \int \csc t dt \\ &= \\ &= \ln\left| \qquad\qquad \right|+C \\ &= \ln\left|\frac{x-1}{\sqrt{x^2+2x-3}}\right|+C\end{aligned}$$

- Integration of "linear over quadratic": Break into two!
  - Example:

$$\begin{aligned}\int \frac{4x+5}{x^2+2x-3}dx &= \int \frac{2(\qquad)+1}{x^2+2x-3}dx \\ &= 2 \qquad\qquad + \int \frac{1}{x^2+2x-3}dx \\ &= 2\int \frac{d(x^2+2x-3)}{x^2+2x-3} + \int \frac{1}{x^2+2x-3}dx \\ &= 2\ln|x^2+2x-3| + \ln\left|\frac{x-1}{\sqrt{x^2+2x-3}}\right|+C\end{aligned}$$

- Other kinds of fractions: Partial fractions!

- Example:

$$\begin{aligned}
\int \frac{x^5+x^3-1}{x^2(x^2+1)^2}dx &= \int \left(\frac{x^5}{x^2(x^2+1)^2} + \frac{x^3}{x^2(x^2+1)^2} - \frac{1}{x^2(x^2+1)^2} \right) dx \\
&= \int \frac{\quad}{(x^2+1)^2}dx + \int \frac{\quad}{(x^2+1)^2}dx - \int \frac{1}{x^2(x^2+1)^2}dx \\
\int \frac{x^3}{(x^2+1)^2}dx &= \int \qquad dx - \int \qquad dx \\
&= \frac{1}{2}\ln\left|x^2+1\right| + \frac{1}{2}\cdot\frac{1}{x^2+1} + C_1 \\
\int \frac{x}{(x^2+1)^2}dx &= \qquad \text{(see above!)} \\
\int \frac{dx}{x^2(x^2+1)^2} &= \int \quad dx + \int \qquad dx + \int \qquad dx \\
&= -\frac{1}{x} - \int \frac{1}{x^2+1}dx - \int \frac{(x^2+1)-x^2}{(x^2+1)^2}dx \\
&= -\frac{1}{x} - \int \frac{1}{x^2+1}dx - \int \frac{1}{x^2+1}dx + \int \frac{x^2}{(x^2+1)^2}dx \\
&=
\end{aligned}$$

$$\begin{aligned}
\therefore \quad \int \frac{x^5+x^3-1}{x^2(x^2+1)^2}dx &= \frac{1}{2}\ln\left|x^2+1\right| + \frac{1}{2}\cdot\frac{1}{x^2+1} + C_1 - \frac{1}{2}\cdot\frac{1}{x^2+1} + C_2 + \frac{1}{x} + \frac{3}{2}\tan^{-1}x + \frac{1}{2}\frac{x}{x^2+1} - C_3 \\
&=
\end{aligned}$$

Exercises

1. Evaluate $\int \frac{x^2+x+1}{x^3-4x^2+x+6}dx$

2. Evaluate $\int \frac{x^2}{(x+1)(x-1)^3} dx$

3. Evaluate $\int \frac{3x^3 - 2x - 20}{(x^2+3)(2x^2 - 6x + 5)} dx$

3 Bring-home Practices

1. $\int \frac{(3x-1)}{x^2+9} dx$
2. $\int \frac{x}{\sqrt{27+6x-x^2}} dx$
3. $\int \frac{x}{(x+1)(x+3)(x+5)} dx$
4. $\int \frac{x^5 - x^3 + 1}{x^4 - x^3} dx$
5. $\int \sec^5 x dx$
6. $\int \tan^5 x dx$
7. $\int \frac{dx}{x\sqrt{x^2+3}}$
8. $\int \frac{xdx}{\sqrt{3+2x-x^2}}$
9. $\int e^{ax} \sin bx dx$