Remedial Lesson 5: Definite Integrals

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1 Definition of Definite Integral

- Definite integral as limit of sum
- Riemann Integral:

$$\int_a^b f(x)dx = \lim_{||x||\to 0} \sum_{k=0}^{n} f(x_k)(x_{k+1} - x_k)$$
$$\text{where} \quad a = x_0 < x_1 < \cdots < x_n = b$$
$$||x|| = \max_k (x_{k+1} - x_k)$$

- Riemann-Stieltjes Integral:

$$\int_a^b f(x)dG(x) = \lim_{||x||\to 0} \sum_{k=0}^{n} f(x_k)[G(x_{k+1}) - G(x_k)]$$
$$\text{where} \quad a = x_0 < x_1 < \cdots < x_n = b$$
$$||x|| = \max_k (x_{k+1} - x_k)$$

$$\int_a^b f(x)dG(x) = \int_a^b f(x)g(x)dx$$

- Newton-Leibniz Formula:

$$\int_a^b f(x)dx = \left[\int f(x)dx\right]_a^b = F(b) - F(a)$$

  - Constant of integration is ignored (as it will be cancelled eventually)

Examples

1. Find the area under the curve $y = x^2$ from $x = 0$ to $x = 1$

$$\int_0^1 x^2 dx$$
$$= \left[\frac{1}{3}x^3\right]_0^1$$
$$= \frac{1}{3}$$

2. Find the are under the curve $y = 2x+1$ from $x = 0$ to $x = 2$

$$\begin{aligned}
&\int_0^2 (2x+1)dx && \text{(trapezeum) Left base} = 1\\
&= \left[\int (2x+1)dx\right]_0^2 && \text{Right base} = 5\\
&= \left[x^2+x\right]_0^2 && \text{Height} = 2\\
&= 2^2+2 && \text{Area} = \frac{1}{2}(1+5)(2)\\
&= 6 && = 6
\end{aligned}$$

3. Find the area of half unit-circle $y = \sqrt{1-x^2}$

$$\begin{aligned}
\min\ x = -1; \quad & \max\ x = +1\\
\therefore\ \text{area} &= \int_{-1}^{1} \sqrt{1-x^2}dx\\
&= \int_{x=-1}^{x=1} \sqrt{1-\sin^2 t}d(\sin t)\\
&= \int_{-\pi/2}^{\pi/2} \cos^2 t dt\\
&= \int_{-\pi/2}^{\pi/2} \frac{1+\cos 2t}{2}dt\\
&= \left[\frac{1}{2}t + \frac{1}{4}\sin 2t\right]_{-\pi/2}^{\pi/2}\\
&= \frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{2}\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}
\end{aligned}$$

Exercises

1. Find the area of the semicircle: $y = \sqrt{r^2-x^2}$

$$\begin{aligned}
\text{Area} &= \int_{-r}^{r} \sqrt{r^2-x^2}dx\\
&= \int_{x=-r}^{x=r} r\cos t\sqrt{r^2-r^2\sin^2 t}dt && \text{(sub } x = r\sin t)\\
&= \int_{-\pi/2}^{\pi/2} r^2\cos^2 t dt\\
&= r^2\int_{-\pi/2}^{\pi/2} \frac{1-\cos 2t}{2}dt\\
&= r^2\left[\frac{1}{2}t + \frac{1}{4}\sin 2t\right]_{-\pi/2}^{\pi/2}\\
&= r^2\left[\frac{1}{2}\frac{\pi}{2} - \frac{1}{2}\left(-\frac{\pi}{2}\right)\right]\\
&= \frac{1}{2}\pi r^2
\end{aligned}$$

2. Find the area between the x-axis and the upper half of the ellipse $a^2x^2 + b^2y^2 = r^2$

$$
\begin{aligned}
a^2x^2 + b^2y^2 &= r^2 \\
y^2 &= \frac{r^2 - a^2x^2}{b^2} \\
y &= \sqrt{\frac{r^2 - a^2x^2}{b^2}} \\
\therefore \quad \min x &= -\frac{r}{a} \\
\max x &= \frac{r}{a} \\
\therefore \text{ area} &= \int_{-r/a}^{r/a} \sqrt{\frac{r^2 - a^2x^2}{b^2}} dx \\
&= \int_{x=-r/a}^{x=r/a} \sqrt{\frac{r^2 - r^2 \sin^2 t}{b^2}} \left(\frac{r}{a} \cos t\right) dt && \left(\text{sub } x = \frac{r}{a} \sin t\right) \\
&= \int_{-\pi/2}^{\pi/2} \frac{r^2}{ab} \cos^2 t dt \\
&= \frac{r^2}{ab} \int_{-\pi/2}^{\pi/2} \frac{1 + \cos 2t}{2} dt \\
&= \frac{r^2}{ab} \left[\frac{1}{2}t + \frac{1}{4} \sin 2t\right]_{-\pi/2}^{\pi/2} \\
&= \frac{r^2}{ab} \left[\frac{1}{2}\frac{\pi}{2} - \frac{1}{2}\left(-\frac{\pi}{2}\right)\right] \\
&= \frac{\pi r^2}{2ab}
\end{aligned}
$$

2 Properties of Definite Integral

- Definite integral is the limit of sum / area under the curve
- Area enclosed by a counter-clockwise path is positive, otherwise it is negative
  - Example: $\int_0^{\pi} \sin x dx > 0$ but $\int_{\pi}^{2\pi} \sin x dx < 0$
  - In terms of the area under the curve $y = f(x)$, the area is positive if $f(x) > 0$, and negative if $f(x) < 0$
  - Integration from right to left reversed the sign

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

- Area equals to the sum of sub-area

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

  - The above formula is true for both $a < b < c$ and $a < c < b$

- Rule of substitution: If $x = g(t)$ is used for substitution,

$$
\begin{aligned}
\int_a^b f(x)dx &= \int_{\alpha}^{\beta} f(g(t))g'(t)dt \\
\text{where } a &= g(\alpha) \\
b &= g(\beta)
\end{aligned}
$$

- Dummy variable: The variable used in definite integral is unimportant,

$$\int_a^b f(x)dx = \int_a^b f(t)dt$$

Exercises

1. Prove $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$\begin{aligned}
\int_0^a f(x)dx &= \int_{x=0}^{x=a} f(a-t)d(a-t) && \text{(sub } x = a-t\text{)} \\
&= -\int_a^0 f(a-t)dt && \begin{cases} d(a-t) & = -dt \\ x = 0 & \implies t = a \\ x = a & \implies t = 0 \end{cases} \\
&= \int_0^a f(a-t)dt && \text{(reverse sign)} \\
&= \int_0^a f(a-x)dx && \text{(dummy variable)}
\end{aligned}$$

2. Prove $\int_0^\pi x f(\sin x)dx = \frac{\pi}{2}\int_0^\pi f(\sin x)dx$ (Hint: Prove their difference is zero)

$$\begin{aligned}
\int_0^\pi x f(\sin x)dx - \frac{\pi}{2}\int_0^\pi f(\sin x)dx &= \int_0^\pi (x - \frac{\pi}{2}) f(\sin x)dx \\
&= \int_0^\pi (\pi - x - \frac{\pi}{2}) f(\sin(\pi - x))dx && \text{(use exercise q.1)} \\
&= \int_0^\pi (\frac{\pi}{2} - x) f(\sin x)dx \\
&= -\int_0^\pi (x - \frac{\pi}{2}) f(\sin x)dx
\end{aligned}$$

$$\because \quad \int_0^\pi (x - \frac{\pi}{2}) f(\sin x)dx = -\int_0^\pi (x - \frac{\pi}{2}) f(\sin x)dx$$

$$\therefore \quad \int_0^\pi (x - \frac{\pi}{2}) f(\sin x)dx = \int_0^\pi x f(\sin x)dx - \frac{\pi}{2}\int_0^\pi f(\sin x)dx = 0$$

3. Evaluate $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx$

$$\begin{aligned}
\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx &= \int_0^\pi \frac{x\sin x}{2-\sin^2 x}dx \\
&= \frac{\pi}{2}\int_0^\pi \frac{\sin x}{2-\sin^2 x}dx && \text{(use exercise q.2)} \\
&= \frac{\pi}{2}\int_{x=0}^{x=\pi} \frac{-1}{1+\cos^2 x}d(\cos x) \\
&= \frac{\pi}{2}\int_{-1}^{1} \frac{1}{1+t^2}dt && \text{(sub } t = \cos x\text{, and reverse of sign)} \\
&= \frac{\pi}{2}\left[\tan^{-1} t\right]_{-1}^{1} \\
&= \frac{\pi}{2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] \\
&= \frac{\pi^2}{4}
\end{aligned}$$

4. Simplify and find the derivative of $g(x) = \dfrac{\sqrt{1+\cos x}}{\sqrt{1+\cos x}+\sqrt{1-\cos x}}$ and hence find $\int_0^{\pi} g(x)dx$

$$\begin{aligned}
g(x) &= \frac{\sqrt{1+\cos x}}{\sqrt{1+\cos x}+\sqrt{1-\cos x}} \\
&= \frac{\sqrt{1+\cos x}}{\sqrt{1+\cos x}+\sqrt{1-\cos x}} \cdot \frac{\sqrt{1+\cos x}-\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}} \\
&= \frac{(1+\cos x)-\sqrt{1-\cos^2 x}}{(1+\cos x)-(1-\cos x)} \\
&= \frac{1+\cos x-\sin x}{2\cos x} \\
\therefore \quad g'(x) &= \frac{2\cos x(-\sin x-\cos x)-(1+\cos x-\sin x)(-2\sin x)}{(2\cos x)^2} \\
&= \frac{(-2\cos x\sin x-2\cos^2 x)+2(\sin x+\sin x\cos x-\sin^2 x)}{4\cos^2 x} \\
&= \frac{2\sin x-2}{4\cos^2 x} \\
&= \frac{\sin x-1}{2\cos^2 x} \\
&= -\frac{1-\sin x}{2(1-\sin^2 x)} \\
&= -\frac{1}{2(1+\sin x)}
\end{aligned}$$

$$\begin{aligned}
\int_0^{\pi} g(x)dx &= \left[xg(x)-\int xg'(x)dx\right]_0^{\pi} && \text{(integration by parts)} \\
&= [xg(x)]_0^{\pi}+\frac{\pi}{2}\int_0^{\pi} g'(x)dx && \text{(using exercise q.2)} \\
&= [xg(x)]_0^{\pi}+\frac{\pi}{2}[g(x)]_0^{\pi} \\
&= 0+\frac{\pi}{2}[g(\pi)-g(0)] \\
&= -\frac{\pi}{2}\left[0-\frac{\sqrt{2}}{\sqrt{2}}\right] \\
&= \frac{\pi}{2}
\end{aligned}$$

3 Helpful Knowledge

- Odd function means for all x, we have $f(-x) = -f(x)$
  - Example: $\sin x$
  - For integration of the odd function, $\int_{-a}^{a} f(x)dx = 0$
- Even function means for all x, we have $f(-x) = f(x)$
  - Example: $\cos x$
  - For integration of the even function, $\int_{-a}^{a} f(x)dx = 2\int_0^{a} f(x)dx$

- Periodic function with period T means for all x, we have $f(x) = f(x+T)$
  - Example: $\sin x$ has period of 2π
  - For integration of the periodic function with period T, we have
    1. $\int_a^b f(x)dx = \int_{a+T}^{b+T} f(x)dx$
    2. $\int_0^T f(x)dx = \int_a^{a+T} f(x)dx$
    3. $\int_0^{nT} f(x)dx = n\int_0^T f(x)dx$

Exercises

1. Evaluate $\int_{-1}^{1}(e^x+e^{-x})\sin x dx$

$$\begin{aligned} \text{Let } f(x) &= (e^x+e^{-x})\sin x \\ \text{and } f(-x) &= -f(x) \\ \therefore \int_{-1}^{1} f(x)dx &= 0 \end{aligned}$$

2. Evaluate $\int_{-1}^{1}(e^x-e^{-x})\cos x dx$

$$\begin{aligned} \text{Let } f(x) &= (e^x-e^{-x})\cos x \\ \text{and } f(-x) &= -f(x) \\ \therefore \int_{-1}^{1} f(x)dx &= 0 \end{aligned}$$

3. Evaluate $\int_{6\pi/7}^{20\pi/7}\sin x dx$

$$\begin{aligned} \int_{6\pi/7}^{20\pi/7}\sin x dx &= \int_0^{2\pi}\sin x dx \\ &= \int_0^{\pi}\sin x dx + \int_{\pi}^{2\pi}\sin x dx = \int_0^{\pi}\sin x dx - \int_0^{\pi}\sin x dx \\ &= 0 \end{aligned}$$

4 Integrated Exercises

1. $\int_0^1 e^{\sqrt{x}}dx$

$$\begin{aligned} \int_0^1 e^{\sqrt{x}}dx &= \int_0^1 e^t(2t)dt && (\text{sub } t=\sqrt{x},\ dx=2tdt) \\ &= 2\int_{t=0}^{t=1} td(e^t) \\ &= 2\left[te^t\right]_0^1 - 2\int_0^1 e^t dt \\ &= 2e-2\left[e^t\right]_0^1 \\ &= 2e-2(e-1) \\ &= 2 \end{aligned}$$

2. $\int_1^2 \frac{e^{2x}}{e^x - 1} dx$

$$\begin{aligned}
\int_1^2 \frac{e^{2x}}{e^x-1}dx &= \int_1^2 \frac{e^x e^x}{e^x-1}dx \\
&= \int_e^{e^2} \frac{e^x}{e^x-1}d(e^x) \\
&= \int_e^{e^2} \frac{tdt}{t-1} = \int_e^{e^2}(1+\frac{1}{t-1})dt \\
&= [t]_e^{e^2} + \int_e^{e^2}\frac{dt}{t-1} \\
&= (e^2-e) + [\ln|t-1|]_e^{e^2} \\
&= (e^2-e) + \ln|e^2-1| - \ln|e-1| \\
&= e^2 - e + \ln\left|\frac{e^2-1}{e-1}\right|
\end{aligned}$$

3. $\int_0^{\pi/3} x\sin 3x dx$

$$\begin{aligned}
\int_0^{\pi/3} x\sin 3x dx &= -\int_{x=0}^{x=\pi/3} xd(\cos 3x) \\
&= -[x\cos 3x]_{x=0}^{x=\pi/3} + \int_0^{\pi/3}\cos 3x dx \\
&= \frac{\pi}{3} + \int_0^{\pi/3}\cos 3x dx \\
&= \frac{\pi}{3} + \left[\frac{1}{3}\sin 3x\right]_0^{\pi/3} \\
&= \frac{\pi}{3} + 0 \\
&= \frac{\pi}{3}
\end{aligned}$$

4. $\int_{-1}^4 f(x)dx$ where $f(x) = \begin{cases} -2x & \text{if } x \le 0 \\ \frac{1}{2}x & \text{if } 0 < x \le 2 \\ 2x-3 & \text{if } x > 2 \end{cases}$

$$\begin{aligned}
\int_{-1}^4 f(x)dx &= \int_{-1}^0 f(x)dx + \int_0^2 f(x)dx + \int_2^4 f(x)dx \\
&= \int_{-1}^0(-2x)dx + \int_0^2(\frac{1}{2}x)dx + \int_2^4(2x-3)dx \\
&= \left[-x^2\right]_{-1}^0 + \left[\frac{1}{4}x^2\right]_0^2 + \left[x^2-3x\right]_2^4 \\
&= [0-(-1)] + [1-0] + [4-(-2)] \\
&= 1+1+6 \\
&= 8
\end{aligned}$$

5. $\int_0^{\infty} xe^{-x} dx$

$$\begin{aligned}\int_0^{\infty} xe^{-x} dx &= -\int_{x=0}^{x=\infty} xd(e^{-x}) \\ &= -\left[xe^{-x}\right]_{x=0}^{\infty} + \int_0^{\infty} e^{-x} dx \\ &= -0 + \left[-e^{-x}\right]_0^{\infty} \\ &= 0 - (-1) \\ &= 1\end{aligned}$$

6. $\int_0^1 \frac{1}{\sqrt{x}} dx$

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x}} dx &= \int_0^1 \frac{2t dt}{t} \qquad \text{(sub } t = \sqrt{x},\ dx = 2t dt\text{)} \\ &= \int_0^1 2dt \\ &= [2t]_0^1 \\ &= 2\end{aligned}$$

7. $\int_0^a \frac{x}{\sqrt{a^2 - x^2}} dx$

$$\begin{aligned}\int_0^a \frac{x}{\sqrt{a^2 - x^2}} dx &= \int_0^{a^2} \frac{d(x^2)}{2\sqrt{a^2 - x^2}} \\ &= -\int_{a^2}^0 \frac{d(a^2 - x^2)}{2\sqrt{a^2 - x^2}} \\ &= \int_0^{a^2} \frac{dt}{2\sqrt{t}} \\ &= \left[\sqrt{t}\right]_0^{a^2} \\ &= a\end{aligned}$$

8. Given $I_n = \int_0^{\pi/4} \sec^n x dx$, Express I_3 in terms of I_1

$$\begin{aligned}I_3 &= \int_0^{\pi/4} \sec^3 x dx \\ &= \int_0^{\pi/4} \sec x (\sec^2 x) dx \\ &= [\sec x \tan x]_0^{\pi/4} - \int_0^{\pi/4} \tan^2 x \sec x dx \\ &= \sqrt{2} - \int_0^{\pi/4} (\sec^2 x - 1) \sec x dx \\ &= \sqrt{2} - \int_0^{\pi/4} \sec^3 x dx + \int_0^{\pi/4} \sec x dx \\ &= \sqrt{2} - I_3 + I_1 \\ \therefore \quad 2I_3 &= \sqrt{2} + I_1 \\ I_3 &= \frac{\sqrt{2}}{2} + \frac{1}{2} I_1\end{aligned}$$