Remedial Lesson 6: Application of Calculus I

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1 Find area using integration

- Given the curve $y = f(x)$ in Cartesian coordinates, the area under the curve from $x = a$ to $x = b$ is given by

$$A = \int_a^b f(x)dx$$

- Given the curve $r = f(\theta)$ in polar coordinates, the area bounded by the curve and the radial vectors $\theta = a$ and $\theta = b$ is given by

$$A = \frac{1}{2}\int_a^b r^2 d\theta$$

- Actually, polar coordinate and Cartesian coordinate can be interchanged:

$$r^2 = x^2 + y^2 \qquad x = r\cos\theta$$
$$\theta = \tan^{-1}\frac{y}{x} \qquad y = r\sin\theta$$

- Example: Find the area of circle with radius r in Cartesian coordinate

$$\begin{aligned}
\text{Equation: } & x^2 + y^2 = r^2 \\
\therefore\quad & y^2 = r^2 - x^2 \\
\therefore\quad & A = 2\int_{-r}^{r}\sqrt{r^2 - x^2}dx \\
& = \qquad\qquad \text{(sub } x = r\sin t\text{)} \\
& = \\
& = \\
& = \\
& = \\
& = \pi r^2
\end{aligned}$$

- Example: Find the area of circle with radius r in Polar coordinate

$$
\begin{aligned}
A &= \frac{1}{2}\int_0^{2\pi} r^2 d\theta \\
&= \\
&= \\
&= \pi r^2
\end{aligned}
$$

- Example: Find $\int_0^\infty e^{-x^2}dx$

$$
\begin{aligned}
&\int_0^\infty e^{-x^2}dx \\
=&\sqrt{\int_0^\infty e^{-x^2}dx \cdot \int_0^\infty e^{-y^2}dy} \\
=&\sqrt{\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy} \\
=&\sqrt{\int_0^\infty \int_0^{\pi/2} e^{-r^2}rd\theta dr} \\
=&\sqrt{\int_0^\infty \int_0^{\pi/2} d\theta e^{-r^2}rdr} \\
=&\sqrt{\frac{\pi}{2}\int_0^\infty re^{-r^2}dr} \\
=&\sqrt{\frac{\pi}{4}\int_0^\infty e^{-r^2}dr^2} \\
=&\sqrt{\frac{\pi}{4}\left[-e^{-r^2}\right]_0^\infty} \\
=&\sqrt{\frac{\pi}{4}[0-(-1)]} \\
=&\frac{\sqrt{\pi}}{2}
\end{aligned}
$$

$$
\begin{cases}
x^2+y^2 &= r^2 \\
dxdy &= \frac{1}{2}d(r^2)d\theta
\end{cases}
$$

(integration of first quadrant)

2 Find limit using L'Hôpital's Rule

- Limit means the value of a function as the variable approaches a value
  - Example: As x tends to 1, $f(x) = x+1$ tends to 2, i.e. $\lim_{x\to 1} f(x) = 2$
  - Example:
$$\lim_{x\to 2}\frac{x^2-4x+4}{x-2} = \lim_{x\to 2}\frac{(x-2)^2}{x-2} = \lim_{x\to 2}(x-2) = 0$$
- We usually interested at the limit towards ∞, $-\infty$, and 0

$$
\begin{aligned}
\lim_{x\to\infty}\frac{x^2+2}{x} &= \infty \\
\lim_{x\to-\infty}\frac{x+1}{x+2} &= 1 \\
\lim_{x\to 0}\frac{x+1}{x^2} &= \infty
\end{aligned}
$$

- Sometimes, we cannot find the limit so easily, so we have the l'Hôpital's rule:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

  this is applicable when the direct substitution have the undeterminate forms: $\frac{\infty}{\infty}$, $\frac{0}{0}$, $0\cdot\infty$, $\infty-\infty$, 0^0, ∞^0, 1^∞

- Example:

$$\begin{aligned}\lim_{x\to 1}\frac{x-1}{\sqrt{x^2-1}} &= \\ \\ &= \\ &= \frac{0}{1} \\ &= 0\end{aligned}$$

- Example:

$$\begin{aligned}\lim_{x\to 0}\frac{\tan x}{\tan 3x} &= \\ &= \end{aligned}$$

- Example:

$$\begin{aligned}\lim_{x\to 0}\tan x\ln x &= \\ &= \\ &= \\ &= \\ &= \end{aligned}$$

- Example:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin x}{x} &= \\ &= \\ &= 1\end{aligned}$$

3 Verify Series Convergence by Integration

- Given the monotonically decreasing function $f(x)$, the infinite series, $\sum_{x=k}^{\infty} f(x)$ is bounded (i.e. not infinitely large), if and only if $\int^{\infty} f(x)dx$ also bounded (evaluate only the upper limit)
- Example:

$$
\begin{aligned}
\frac{1}{x} &> \frac{1}{x+1} && \therefore \text{ decreasing} \\
\int^{\infty} \frac{1}{x} dx &= [\ln x]^{\infty} \\
&= \ln \infty \\
&= \infty \\
\therefore \quad \sum_{x=1}^{\infty} \frac{1}{x} &= \infty && \text{i.e. diverging series}
\end{aligned}
$$

- Example:

$$
\begin{aligned}
\frac{1}{x^2} &> \frac{1}{(x+1)^2} && \therefore \text{ decreasing} \\
\int^{\infty} \frac{1}{x^2} dx &= \left[\frac{-1}{x}\right]^{\infty} \\
&= 0 \\
&< \infty \\
\therefore \quad \sum_{x=1}^{\infty} \frac{1}{x^2} &< \infty && \text{i.e. converging}
\end{aligned}
$$

- Example: Check for the convergence of $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$

$$
\begin{aligned}
\int^{\infty} \frac{x^2}{x^3+1} dx &= \int^{\infty} \frac{d(x^3)}{3(x^3+1)} \\
&= \\
&= \infty \\
\therefore \quad \sum_{n=1}^{\infty} \frac{n^2}{n^3+1} &= \infty \quad \text{(diverging)}
\end{aligned}
$$

- Example: Check for the convergence of $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+9}}$

4 Approximation using Differentials

- Differential:

$$dy = f(x)dx$$

- Hence we can approximate the derivation of y by

$$\begin{aligned} f(x+\Delta x) &= y + \Delta y \\ &\approx f(x) + f'(x)\Delta x \end{aligned}$$

- This is the basis for "small perturbation analysis" and why we need to study linear systems in detail
- Example: Find a **good** approximate of $\sqrt{4.1}$ without using calculator

$$\begin{aligned} \frac{d}{dx}\sqrt{x} &= \frac{1}{2\sqrt{x}} \\ \therefore \quad \sqrt{4+0.1} &\approx \sqrt{4} + \frac{1}{2\sqrt{4}}(0.1) \\ &= 2 + \frac{1}{4}(0.1) \\ &= 2.025 \\ \text{Actually, } \sqrt{4.1} &= 2.02484567\ldots \end{aligned}$$

- Example: Find a good approximate of $\sqrt[3]{7}$ without using calculator

- Example: Find a good approximate of π^2

5 Using integration to solve differential equations

- Differential equations is the equation involving derivatives of functions
- Example:

$$f(x)+\frac{d}{dx}f(x)=x^2+2x$$

- Solving differential equation means finding out the function, for example, the solution for the above equation is $f(x)=x^2$.
- The easiest form of differential equation is the separable equation, namely, we can write the equation in the form:

$$g(y)\frac{dy}{dx}=f(x)$$

which can be solved by:

$$\begin{aligned} g(y)\frac{dy}{dx} &= f(x) \\ g(y)dy &= f(x)dx \\ \therefore \quad \int g(y)dy &= \int f(x)dx \end{aligned}$$

- Example: Solve y for $\frac{dy}{dx}=\frac{y^2}{1+x^2}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{y^2}{1+x^2} \\ \therefore \quad \frac{1}{y^2}dy &= \frac{1}{1+x^2}dx \\ \int \frac{1}{y^2}dy &= \int \frac{1}{1+x^2}dx \\ -\frac{1}{y} &= \tan^{-1}x+C \\ y &= \frac{-1}{\tan^{-1}x+C} \end{aligned}$$

- Example: Solve y for $\frac{dy}{dx}=\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$, given $y=1$ when $x=0$

- Example: A stationary particle of mass m fall under gravity. When it has velocity v, it experiences a resistance force $f(v) = -2v$. Express displacement s in terms of time t.

6 Maclaurin Series and Taylor Series

- Maclaurin Series is to express *any* function $f(x)$ as the infinite power series:

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{x^k}{k!} f^{(k)}(0) \\ &= f(0) + x f'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + \cdots \end{aligned}$$

- Taylor Series is a generalization of Maclaurin Series:

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!} f^{(k)}(a) \\ &= f(a) + (x-a) f'(0) + \frac{(x-a)^2}{2} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots + \frac{(x-a)^n}{n!} f^{(n)}(a) + \cdots \end{aligned}$$

  so we usually call Maclaurin series as Taylor series.

- With Taylor series, everything can be expressed as polynomial

- Example: Express e^x as Taylor series

$$\begin{aligned}
e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} f^{(k)}(0) \\
&= \sum_{k=0}^{\infty} \frac{x^k}{k!} e^0 \\
&= \sum_{k=0}^{\infty} \frac{x^k}{k!} \\
&= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots
\end{aligned}$$

- Example: Express $\sin x$ as Taylor series

- Example: Express $\cos x$ as Taylor series

- Example: Express $\frac{1}{x+1}$ as Taylor series