Remedial Lesson 7: Application of Calculus II

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October 10, 2005

1 FYI: Laplace Transform and Fourier Transform

- Laplace Transform:

$$\begin{aligned} F(s) = \mathscr{L}\{f(t)\} &= \int_0^\infty e^{-st} f(t)dt \\ f(t) = \mathscr{L}^{-1}\{F(s)\} &= \lim_{R\to\infty} \frac{1}{2\pi i} \int_{\beta-iR}^{\beta+iR} e^{st} F(s)ds \end{aligned}$$

- Laplace transform is a function of functions, i.e.
  - Input: a function in t
  - Output: a function in s
  - For nearly any function in t, we can find an unique corresponding function in s
  - If we get the output, we can revert and get back the input (may differ by a constant)
- Use: Solving differential equations
- Table of Laplace Transform:

$f(t)$	$F(s)$
$f(t)$	$\int_0^\infty e^{-st} f(t)dt$
$af(t)+bg(t)$	$aF(s)+bG(s)$
$e^{at}f(t)$	$F(s-a)$
$f(t-a)u(t-a)$	$e^{-as}F(s)$
$f(at)$	$\frac{1}{a}F(\frac{s}{a})$
$f^{(n)}(t)$	$s^nF(s)-\sum_{k=0}^{n-1} s^{n-1-k}f^{(k)}(0)$
$\int_0^t f(\tau)d\tau$	$\frac{1}{s}F(s)$
$tf(t)$	$-F'(s)$
$\frac{1}{t}f(t)$	$\int_s^\infty F(\sigma)d\sigma$
$f(t)*g(t)$	$F(s)G(s)$
$f(0)$	$\lim_{s\to\infty} sF(s)$
$f(\infty)$	$\lim_{s\to 0} sF(s)$

$f(t)$	$F(s)$	$f(t)$	$F(s)$
t	$\frac{1}{s^2}$	$\delta(t)$	1
t^2	$\frac{2}{s^3}$	1 or $u(t)$	$\frac{1}{s}$
$t^n,\ n=0,1,2,\dots$	$\frac{n!}{s^{n+1}}$	$u(t-a)$	$\frac{1}{s}e^{-as}$
e^{at}	$\frac{1}{s-a}$	$\delta(t-a)$	e^{-as}
$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$	te^{-at}	$\frac{1}{(s+a)^2}$
$\cos\omega t$	$\frac{s}{s^2+\omega^2}$	$t\sin\omega t$	$\frac{2\omega s}{(s^2+\omega^2)^2}$
$\sinh\omega t$	$\frac{\omega}{s^2-\omega^2}$	$e^{at}\sin\omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$
$\cosh\omega t$	$\frac{s}{s^2-\omega^2}$	$e^{at}\cos\omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
$f'(t)$	$sF(s)-f(0)$		
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$		
$tf'(t)$	$-F(s)-sF'(s)$		
$tf''(t)$	$-2sF(s)-s^2F'(s)-f(0)$		

- Fourier transform:

$$F(\omega) = \mathscr{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$
$$f(x) = \mathscr{F}^{-1}\{F(\omega)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega$$

- Fourier transform is also a function of functions, i.e.
  - Input: a function in time domain, t
  - Output: a function in frequency domain, f
- Use: Frequency domain analysis
- Table of fourier transform

$f(t)$	$F(\omega)$	$f(t)$	$F(\omega)$
$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$	$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$	$\delta(t)$	1
$af(t) + bg(t)$	$aF(\omega) + bG(\omega)$	$e^{i\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$
$e^{i\omega_0 t} f(t)$	$F(\omega - \omega_0)$	$u(t)$	$\pi\delta(\omega) + \frac{1}{i\omega}$
$f(t - t_0)$	$e^{-i\omega t_0} F(\omega)$	$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
$f(at)$	$\frac{1}{a} F(\frac{\omega}{a})$	$\sin \omega_0 t$	$-i\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
$F(t)$	$2\pi f(-\omega)$	$u(t) \cos \omega_0 t$	$\frac{\pi}{2}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \frac{i\omega}{\omega_0^2 - \omega^2}$
$f^{(n)}(t)$	$(i\omega)^n F(\omega)$	$u(t) \sin \omega_0 t$	$\frac{-i\pi}{2}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] + \frac{\omega^2}{\omega_0^2 - \omega^2}$
$(-it)^n f(t)$	$F^{(n)}(\omega)$	$u(t) e^{-at} \cos \omega_0 t$	$\frac{a + i\omega}{\omega_0^2 + (a + i\omega)^2}$
$\int_{-\infty}^{t} f(\tau) d\tau$	$\frac{1}{i\omega} F(\omega) + \pi F(0) \delta(\omega)$	$u(t) e^{-at} \sin \omega_0 t$	$\frac{\omega_0}{\omega_0^2 + (a + i\omega)^2}$
$f(t) * g(t)$	$\sqrt{2\pi} F(\omega) G(\omega)$	$u(t) e^{-at}$	$\frac{1}{a + i\omega}$
$f(t) g(t)$	$\frac{1}{2\pi} F(\omega) * G(\omega)$	$u(t) t e^{-at}$	$\frac{1}{(a + i\omega)^2}$

2 Leibniz's Rule for Order-n Differentiation

- Given function $u(x) = f(x)g(x)$,

$$u'(x) = \frac{d}{dx} u(x) = f(x) \frac{d}{dx} g(x) + g(x) \frac{d}{dx} f(x)$$
$$u''(x) = \frac{d^2}{dx^2} u(x) = f(x) \frac{d^2}{dx^2} g(x) + 2 \frac{d}{dx} f(x) \frac{d}{dx} g(x) + g(x) \frac{d^2}{dx^2} f(x)$$
$$\vdots$$
$$u^{(n)}(x) = \frac{d^n}{dx^n} u(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{d^{n-k}}{dx^{n-k}} f(x) \frac{d^k}{dx^k} g(x)$$

this is called the Leibniz's rule. Which as the form similar to binomial theorem:

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$
$$\frac{d^n}{dx^n} (f(x) \cdot g(x)) = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) g^{(k)}(x)$$

- Example:

$$
\begin{aligned}
y &= x^2 \sin x \\
\frac{d}{dx}\sin x &= \sin(x+\frac{\pi}{2}) \\
\frac{d^2}{dx^2}\sin x &= \sin(x+\frac{2\pi}{2}) \\
&\vdots \\
\frac{d^n}{dx^n}\sin x &= \sin(x+\frac{n\pi}{2}) \\
\frac{d}{dx}x^2 &= 2x \\
\frac{d^2}{dx^2}x^2 &= 2 \\
&\vdots \\
\frac{d^n}{dx^n}x^2 &= \begin{cases} 2x & : n=1 \\ 2 & : n=2 \\ 0 & : \text{otherwise} \end{cases} \\
\therefore \quad \frac{d^{80}}{dx^{80}}x^2\sin x &= \binom{80}{0}x^2\sin(x+\frac{80\pi}{2}) + \binom{80}{1}2x\sin(x+\frac{79\pi}{2}) + \binom{80}{2}2\sin(x+\frac{78\pi}{2}) \\
&= x^2\sin x - 160x\cos x - 6320\sin x
\end{aligned}
$$

Exercises

1. Find the nth derivative of $y = x^3e^{ax}$, $n \geq 3$

$$
\begin{aligned}
\frac{d^n}{dx^n}(x^3e^{ax}) &= \sum_{k=0}^{n}\binom{n}{k}\frac{d^{n-k}}{dx^{n-k}}e^{ax}\frac{d^k}{dx^k}x^3 \\
&= \binom{n}{0}x^3a^ne^{ax} + \binom{n}{1}3x^2a^{n-1}e^{ax} + \binom{n}{2}6xa^{n-2}e^{ax} + \binom{n}{3}6a^{n-3}e^{ax} \\
&= x^3a^ne^{ax} + n3x^2a^{n-1}e^{ax} + \frac{n(n-1)}{2!}6xa^{n-2}e^{ax} + \frac{n(n-1)(n-2)}{3!}6a^{n-3}e^{ax} \\
&= x^3a^ne^{ax} + n3x^2a^{n-1}e^{ax} + 3n(n-1)xa^{n-2}e^{ax} + n(n-1)(n-2)a^{n-3}e^{ax} \\
&= e^{ax}[x^3a^n + n3x^2a^{n-1} + 3n(n-1)xa^{n-2} + n(n-1)(n-2)a^{n-3}]
\end{aligned}
$$

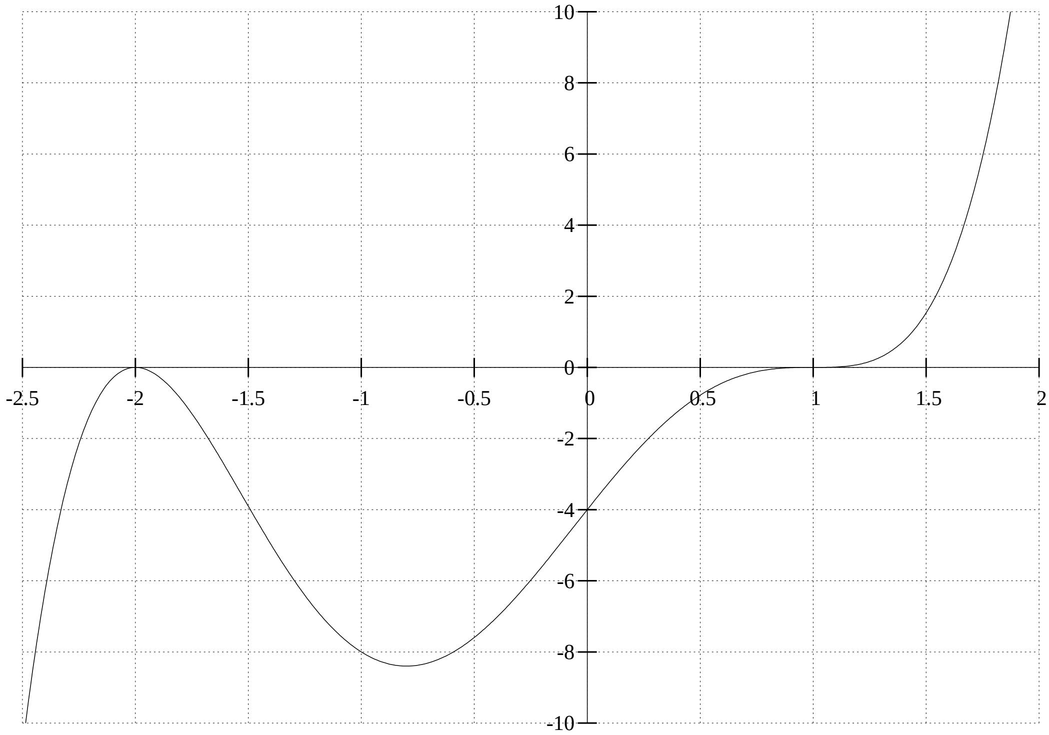
2. Find the nth derivative of $y = 2^x \ln x$

$$
\begin{aligned}
\frac{d^n}{dx^n}2^x\ln x &= \sum_{k=0}^{n}\binom{n}{k}\frac{d^{n-k}}{dx^{n-k}}2^x\frac{d^k}{dx^k}\ln x \\
&= \sum_{k=0}^{n}\binom{n}{k}\frac{d^{n-k}}{dx^{n-k}}e^{x\ln 2}\frac{d^k}{dx^k}\ln x \\
&= e^{x\ln 2}(\ln 2)^n\ln x + \sum_{k=1}^{n}\binom{n}{k}\frac{d^{n-k}}{dx^{n-k}}e^{x\ln 2}\frac{d^k}{dx^k}\ln x \\
&= e^{x\ln 2}(\ln 2)^n\ln x + \sum_{k=1}^{n}\binom{n}{k}\left[e^{x\ln 2}(\ln 2)^{n-k}\right]\cdot\left[(-1)^{k-1}\frac{(k-1)!}{x^k}\right] \\
&= 2^x\left[(\ln 2)^n\ln x + \sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}(\ln 2)^{n-k}\frac{(k-1)!}{x^k}\right]
\end{aligned}
$$

3 Finding Local Extrema

- Given a curve $y = f(x)$, if the point (x', y') is the maximum or minimum of the curve, the slope at that point must be zero, i.e.
$$\frac{d}{dx} f(x)\bigg|_{x=x'} = 0$$
- If it is maximum, the slope of $y = f(x)$ should be decreasing (from positive slope, to zero, to negative), but if it is minimum, the slope of $y = f(x)$ should be increasing (from negative slope, to zero, to positive).
- Point of inflexion is the point (x', y') that gives a zero slope, but it is neither maximum nor minimum
- Example: Find the maximum and minimum values of $f(x) = (x+2)^2(x-1)^3$

$$\begin{aligned}
f(x) &= (x+2)^2(x-1)^3 \\
f'(x) &= 2(x+2)(x-1)^3 + 3(x+2)^2(x-1)^2 \\
&= (x+2)(x-1)^2(5x+4) \\
\therefore\ f'(x) = 0 &\implies x = -2, 1, -\frac{4}{5} \\
x < -2 &\implies f'(x) > 0 \\
-2 < x < -\frac{4}{5} &\implies f'(x) < 0 \\
-\frac{4}{5} < x < 1 &\implies f'(x) > 0 \\
x > 1 &\implies f'(x) > 0 \\
\therefore\ \text{minimum at } x = -\frac{4}{5} &\text{ at } \left(-\frac{4}{5}, -\frac{26244}{3125}\right) \\
\text{maximum at } x = -2 &\text{ at } (-2, 0) \\
\text{inflexion at } x = 1 &\text{ at } (1, 0)
\end{aligned}$$



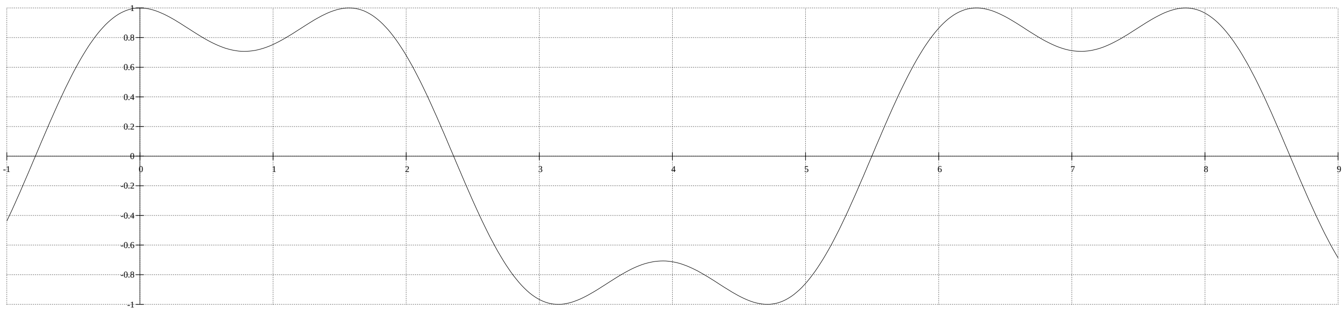
Exercises

1. Find the maximum and minimum values of $f(x) = \sin^3 x + \cos^3 x$

$$\begin{aligned} f(x) &= \sin^3 x + \cos^3 x \\ f'(x) &= 3\sin^2 x \cos x - 3\cos^2 x \sin x \\ &= 3\sin x \cos x(\sin x - \cos x) \\ f'(x) = 0 &\implies x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, 2\pi, \cdots \end{aligned}$$

$$\begin{aligned} \text{By checking, minima at: } & x = \frac{\pi}{4}, \pi, \frac{3\pi}{2} \\ \text{maxima at: } & x = 0, \frac{\pi}{2}, \frac{5\pi}{4}, 2\pi \end{aligned}$$

$$\begin{aligned} \text{with the minima points: } & (\frac{\pi}{4}, \frac{\sqrt{2}}{2})\ (\pi, -1)\ (\frac{3\pi}{2}, -1) \\ \text{maxima points: } & (0,1)\ (\frac{\pi}{2}, 1)\ (\frac{5\pi}{4}, 1)\ (2\pi, 1) \\ & \text{and periodic with period } 2\pi \end{aligned}$$



2. Find the values of x of the function $f(x) = \dfrac{x^2 - 5x + 6}{x^2 + 1}$

$$
\begin{aligned}
f(x) &= \frac{x^2-5x+6}{x^2+1} \\
f'(x) &= \frac{(x^2+1)(2x-5)-(x^2-5x+6)(2x)}{(x^2+1)^2} \\
&= \frac{5x^2-10x-5}{(x^2+1)^2} \\
&= \frac{5(x^2-2x-1)}{(x^2+1)^2} \\
f'(x)=0 &\implies x = 1\pm\sqrt{2}
\end{aligned}
$$

By checking, minima at: $x = 1+\sqrt{2}$

maxima at: $x = 1-\sqrt{2}$

with the minima points: $(1+\sqrt{2}, \frac{4-3\sqrt{2}}{4+2\sqrt{2}})$

maxima points: $(1-\sqrt{2}, \frac{4+3\sqrt{2}}{4-2\sqrt{2}})$