ERG2011A Tutorial 2b: Vector Differentiation

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1 Gradient (of scalar field)

- How to convert a scalar function into vector function?
  Topic of interest: Scalar functions like $f(x, y, z)$
  - Field: (x, y, z)
  - Scalar field: $u = f(x, y, z)$, i.e. every point (x, y, z) correspond to some value
    (think: Electric field, Magnetic field)
- Gradient of a scalar function $f(x, y, z)$ is a (   ) function defined to be:

$$\operatorname{grad} f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

  - Symbol introduced: ∇ (   ) or *del* (caution: ∇ is just an operator similar to $\frac{d}{dx}$!)

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

  - Alternative notation: $\operatorname{grad} f = \nabla f$
  - Gradient is a vector value
  - Physical meaning of ∇f: The direction of most rapid change of $f(x, y, z)$, i.e., maximal increase
- Example: Problem Set 8.9, Question 14
  If on a mountain the elevation above sea level is $z(x, y) = 1500 - 3x^2 - 5y^2$ [meters], what is the direction of steepest ascent at $P : (-0.2, 0.1)$?
  - Steepest ascent at any point $(x, y) = \operatorname{grad} z =$
  - Given point $P = (-0.2, 0.1)$, so we substitute into $\operatorname{grad} z$ and hence

$$\operatorname{grad} z|_{(-0.2,0.1)} =$$

    which is the steepest ascent at P.
- Why we call ∇f the gradient?
  With ∇f, we can get the **directional derivative**, which is the (   ) along a specified direction
  - Denoted by:
- Example: Problem Set 8.9, Question 30
  Find the directional derivative of $f = x - y$ at $P : (4, 5)$ in the direction of $\mathbf{a} = 2\mathbf{i} + \mathbf{j}$.

- Gradient: $\mathrm{grad}\, f = \mathbf{i} - \mathbf{j}$
  since $\mathrm{grad}\, f$ is independent of x and y, it is a constant gradient at all points P
- Direction *unit* vector: $\hat{\mathbf{a}} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{1}{\phantom{\sqrt{5}}}(2\mathbf{i} + \mathbf{j})$
- Directional derivative:

$$
\begin{aligned}
D_{\mathbf{a}} f &= \\
&= \\
&= \frac{2-1}{\sqrt{5}} = \frac{1}{\sqrt{5}}
\end{aligned}
$$

2 Divergence (of vector field)

- Similar to gradient of a scalar function, we have divergence of a vector function $\mathbf{v}(x, y, z) = v_x(x, y, z)\mathbf{i} + v_y(x, y, z)\mathbf{j} + v_z(x, y, z)\mathbf{k}$ defined to be:

$$\mathrm{div}\, \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

  - Alternative notation: $\mathrm{div}\, \mathbf{v} = \nabla \cdot \mathbf{v}$
  - Divergence is a scalar value
  - Application: Gravitational field

- Example of calculating divergence: Problem Set 8.10 Question 8
  - $\mathbf{v} = xyz(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$
  - Divergence:

$$
\begin{aligned}
\mathrm{div}\, \mathbf{v} &= \nabla \cdot \mathbf{v} \\
&= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \right) \cdot \\
&=
\end{aligned}
$$

- Divergence is for (   ) fields, i.e. (   ) -valued functions. Hence if we have a scalar-valued function, we do not have divergence defined.
  - But we have Laplacian of a scalar-valued function defined to be:

$$\mathrm{div}(\mathrm{grad}\, f) = \nabla \cdot (\nabla f) = \nabla^2 f$$

  - If you don't have Laplacian, you can get it by direct differentiation:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

    but this is a nightmare to do.

- Example: Problem Set 8.10 Question 16
  - Function (scalar-valued): $f(x,y) = e^{2x}\sin(2y)$
  - Using Laplacian:

$$\begin{aligned}\nabla^2 f &= \nabla\cdot(\nabla f) = \nabla\cdot\big(\nabla(e^{2x}\sin 2y)\big)\\ &= \nabla\cdot \\ &= \end{aligned}$$

  - Using direct differentiation:

$$\begin{aligned}\nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\\ &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(e^{2x}\sin 2y)\right) + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}(e^{2x}\sin 2y)\right)\\ &= \frac{\partial}{\partial x}(\qquad\qquad) + \frac{\partial}{\partial y}(\qquad\qquad)\\ &= \\ &= 0\end{aligned}$$

  - Same result, but using Laplacian is easier.

3 Curl (of vector field)

- Divergence is $\nabla\cdot\mathbf{v}$
- Curl is $\nabla\times\mathbf{v}$, and it is defined to be

$$\begin{aligned}\operatorname{curl}\mathbf{v} &= \nabla\times\mathbf{v}\\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ v_x & v_y & v_z\end{vmatrix}\\ &= \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\mathbf{k}\end{aligned}$$

  - For any differentiable scalar function, we have $\operatorname{curl}(\operatorname{grad} f) = \mathbf{0}$
  - For any differentiable vector function, we have $\operatorname{div}(\operatorname{curl}\mathbf{v}) = 0$

- Example of calculating curl: Problem Set 8.11 Question 6
  - $\mathbf{v} = [\sin y, \cos z, 0]$
  - $\operatorname{curl} \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y & \cos z & 0 \end{vmatrix}$

    $\operatorname{curl} \mathbf{v} = \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}\cos z\right)\mathbf{i} + \left(\frac{\partial}{\partial z}\sin y - \frac{\partial}{\partial x}(0)\right)\mathbf{j} + \left(\frac{\partial}{\partial x}\cos z - \frac{\partial}{\partial y}\sin y\right)\mathbf{k}$
  - Hence,

    $\nabla \times \mathbf{v} =$

4 Summary:

	grad	div	curl
Notation	∇f	$\nabla \cdot \mathbf{v}$	$\nabla \times \mathbf{v}$
Value	Vector	Scalar	Vector

- What's the use? Let's see the Maxwell's Equations:
  (Reference from http://en.wikipedia.org/wiki/Maxwell's_equations)

$$\begin{array}{lrcl} \text{Gauss' Law} & \nabla \cdot \epsilon \mathbf{E} & = & \rho \\ \text{Gauss' Law for Magnetism} & \nabla \cdot \mathbf{B} & = & 0 \\ \text{Faraday's Law of Induction} & \nabla \times \mathbf{E} & = & -\frac{\partial \mathbf{B}}{\partial t} \\ \text{Ampere's Law} & \nabla \times \frac{1}{\mu}\mathbf{B} & = & \mathbf{J} + \epsilon \frac{\partial \mathbf{E}}{\partial t} \end{array}$$

  where:
  - ρ is the free electric charge density in Cm^{-3}, in vacuum $\rho = 0$
  - ϵ is the electrical permittivity
  - μ is the magnetic permeability
  - $\mathbf{B}$ is the magnetic flux density in tesla, T
  - $\mathbf{E}$ is the electric field in Vm^{-1}
  - $\mathbf{J}$ is the current density in Am^{-2}
- (Maxwell's Equations are not included in ERG2011A, actually)