ERG2011A Tutorial 3: Vector Integration

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1 Line Integral

- Notation of line integral:

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$$

  - Summing up all the vectors from function $\mathbf{F}(\mathbf{r})$ where $\mathbf{r}$ is a vector parameter (e.g. a point in space) supplied to $\mathbf{F}$
  - The summation is adding all the $\mathbf{F} \cdot \mathbf{r}$ such that $\mathbf{r}$ is sweeping the curve C

- Meaning of the line integral:

$$\begin{aligned} \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} dt \\ &\approx \sum_{a \le t \le b} \mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r} \\ &\approx \sum_{a \le t \le b} \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \end{aligned}$$

  - We cannot integrate for $\mathbf{r}$ sweeping (   ) because we cannot represent it mathematically
  - Instead, we represent $\mathbf{r}$ as a (   ) of t, and then when
    * $t = a$, $\mathbf{r}$ is the (   ) point of C
    * $t = b$, $\mathbf{r}$ is the (   ) point of C
    * $a < t < b$, $\mathbf{r}$ is sweeping (   )
  - Example of use: Work done of motion in non-straight line

- No matter how "fast" or how "slow" your ( ) are sweeping, the result is just C
  i.e. *The value of the line integral does not depend on the choice of representation of C*
- But the line integral ( ) depend on the actual path of C

1.1 Calculation

- Representing $\mathbf{F}(\mathbf{r}(t))$ as:
  - Components of $\mathbf{F}$ along x-, y-, and z-axes
- Representing $\mathbf{r}(t)$ as:
  - A moving point, as a parameter supplied to $\mathbf{F}$
- Then we have $d\mathbf{r} = (dx,\ dy,\ dz)$, and then

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot d\mathbf{r}(t) \\ &= \int_a^b (F_1,\ F_2,\ F_3)\cdot (\qquad\qquad) \\ &= \int_a^b (F_1 dx + F_2 dy + F_3 dz)\end{aligned}$$

- Also,

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot \frac{d\mathbf{r}}{dt}dt \\ &= \int_a^b (F_1,\ F_2,\ F_3)\cdot (\qquad\qquad)dt \\ &= \int_a^b \left[F_1 x'(t) + F_2 y'(t) + F_3 z'(t)\right] dt\end{aligned}$$

- Example: Problem Set 9.1 Question 8
  Find $\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r}$ for $\mathbf{F} = [x-y,\ y-z,\ z-x]$ and C defined as the locus of $\mathbf{r} = [2\cos t,\ t,\ 2\sin t]$ from $(2,0,0)$ to $(2,2\pi,0)$
  - Step 1: We try to use $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt$
    * $\mathbf{r} = [2\cos t,\ t,\ 2\sin t]$
    * $(2,0,0) \implies \mathbf{r}(\quad)$, i.e. $t = \quad = a$
    * $(2,2\pi,0) \implies \mathbf{r}(\quad)$, i.e. $t = \quad = b$
  - Step 2:

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_a^b \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt = \int_a^b \left[F_1 x'(t) + F_2 y'(t) + F_3 z'(t)\right] dt \\ &= \int_0 \left[(x-y)(\qquad) + (y-z)(\qquad) + (z-x)(\qquad)\right] dt \\ &= \int_0^{2\pi} \left[(\qquad\qquad)\cdot(-2\sin t) + (\qquad\qquad) + (\qquad\qquad)\cdot 2\cos t\right] dt \\ &= \int_0^{2\pi} \left[-2t\sin t + t - 2\sin t - 4\cos^2 t\right] dt \\ &= -2\left[\sin t - t\cos t\right]_0^{2\pi} + \left[\frac{1}{2}t^2\right]_0^{2\pi} - 2\left[-\cos t\right]_0^{2\pi} - 4\left[\frac{1}{2}t + \frac{1}{4}\sin 2t\right]_0^{2\pi} = 2\pi^2\end{aligned}$$

2 Path-independent Line Integrals

- Remember: Line integral *may* depend on the actual path of C
  - When will it depend, and when will it independent?
- Theorem 1:
  Line integral $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ is independent of the path in domain D if and only if $\mathbf{F}$ is a gradient of some function f in D, i.e. $\mathbf{F} = \operatorname{grad} f$.
  (For proof, read book page 472)
  - The example given in page 1 is a path-independent line integral
    - Because it is a *potential enrgy* problem
  - If the line integral is path-independent, we have

$$\int_{\mathbf{A}}^{\mathbf{B}} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = f(\mathbf{B}) - f(\mathbf{A})$$

    where $\mathbf{A}$ and $\mathbf{B}$ are the initial and terminal points of curve C and $\mathbf{F} = \operatorname{grad} f$.
  - In some applications, we call f the ( ) of $\mathbf{F}$.
    In other words, the line integral is independent of path in D if and only if $\mathbf{F}$ is the gradient of a potential in D.
- Theorem 2:
  Line integral $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ is independent of path in domain D if and only if $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = 0$ for all closed path C in D.
  (For proof, read book page 473 — but it is intuitive)
- Theorem 3:
  Line integral $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ is independent of the path in domain D if and only if $\operatorname{curl} \mathbf{F} = \mathbf{0}$
  - Implies: Differential form of $\mathbf{F} \cdot d\mathbf{r}$ is ( ), i.e.

$$F_1 dx + F_2 dy + F_3 dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = df$$

    or equivalently,

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

- Example: Problem Set 9.2, Question 8
  Show that the form under the integral sign in exact in space and evaluate the integral:
  $\int_{(\pi,\pi/2,2)}^{(0,\pi,1)} (-z \sin xz dx + \cos y dy - x \sin xz dz)$
  - Show:
    - $F_1 dx + F_2 dy + F_3 dz = -z \sin xz dx + \cos y dy - x \sin xz dz$,
      therefore $F_1 = -z \sin xz$, $F_2 = \cos y$, $F_3 = -x \sin xz$
    - $\dfrac{\partial F_3}{\partial y} = 0$, $\dfrac{\partial F_2}{\partial z} = 0$
    - $\dfrac{\partial F_1}{\partial z} = -\sin xz - xz \cos xz$, $\dfrac{\partial F_3}{\partial x} =$
    - $\dfrac{\partial F_2}{\partial x} = 0$, $\dfrac{\partial F_1}{\partial y} = 0$
    - We have shown that $\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}$, $\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$, $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$ and hence it is exact.

- Evaluate:
  * It is path independent, thus the value of the integral is $f(0, \pi, 1) - f(\pi, \pi/2, 2)$.
  * Finding f:

$$\begin{aligned}
f &= \int F_1 dx = \int F_2 dy = \int F_3 dz \\
\int F_1 dx &= \int (-z \sin xz) dx \\
&= \cos xz + g(y, z) \qquad \text{*here's } g(y,z) \text{ is a "constant of integration"}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial y} &= 0 + \frac{\partial g}{\partial y} = \\
g(y, z) &= \int F_2 dy = \int \cos y dy \\
&= \qquad +h(z)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial z} &= -x \sin xz + 0 + \frac{\partial h}{\partial z} = \\
h(z) &=
\end{aligned}$$

$$f = \cos xz + \sin y + c \quad \text{for some constant of integration } c$$

- Subtraction:

$$\begin{aligned}
f(0, \pi, 1) - f(\pi, \pi/2, 2) &= [\cos(0 \cdot 1) + \sin \pi] - [\cos 2\pi + \sin \frac{\pi}{2}] \\
&= \\
&= -1
\end{aligned}$$

3 Double Integrals

- Double Integrals $\neq$ Repeated integral or Iterated integral
- Double integral:

$$\iint_R f(x,y)dxdy$$

  - Adding $f(x,y)$ for all the (x,y) in the region R
  - The differential $dxdy$ is a whole to mean a tiny area in region R
  - Use: Finding center of gravity, Finding volume of arbitrary body

- Evaluation of double integral: Using interated integrals

$$\iint_R f(x,y)dxdy = \int_a^b \left[\int_{g(x)}^{h(x)} f(x,y)dy\right] dx = \int_c^d \left[\int_{p(y)}^{q(y)} f(x,y)dx\right] dy$$

  - Using function $g(x)$ and $h(x)$ to draw a boundary for $f(x,y)$, as follows:

- Techniques: Change of variable in double integral:

$$\iint_R f(x,y)dxdy = \iint_{R'} f(x(u,v),y(u,v)) \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} dudv$$

  - R becomes R', but they are the same region in different ( )
  - The area $dxdy$ in original domain system becomes in the new domain
  - The determinant, $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ is called the ( )
  - Example of use: Polar coordinate sytem $\leftrightarrow$ Cartesian coordinate system

- Example: Problem Set 9.3 Question 10
  Describe the region of integration and evaluate $\int_0^{\pi/4}\int_0^{\cos y} x^2 \sin y dxdy$
  - y is from 0 to $\pi/4$
  - x is from 0 to $\cos y$, i.e. $p(y) = 0$ and $q(y) = \cos y$
  - The region of integration is therefore $\frac{1}{8}$ cycle of the cosine curve, i.e. the area bounded by x- and y-axes and the curve $x = \cos y$

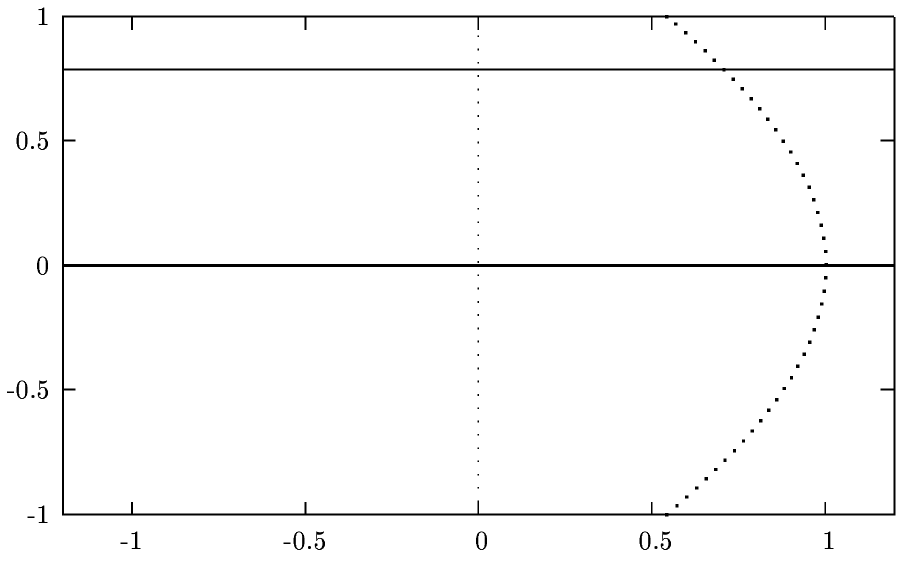

Figure 1: Region of Integration

– Evaluation:

$$\begin{aligned}\int_0^{\pi/4}\int_0^{\cos y} x^2 \sin y dx dy &= \int_0^{\pi/4} \sin y \left(\int_0^{\cos y} x^2 dx\right) dy \\ &= \int_0^{\pi/4} \sin y \left(\frac{1}{3}\cos^3 y\right) dy \\ &= \frac{1}{3}\int_0^{\pi/4} \sin y \cos^3 y dy \\ &= \frac{1}{24}\left[-\cos 2y - \frac{\cos 4y}{4}\right]_0^{\pi/4} \\ &= \frac{1}{16}\end{aligned}$$

4 Summary of formula

Line integral: $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b \left[F_1 x'(t) + F_2 y'(t) + F_3 z'(t)\right] dt$

Path-independent Line integral: $\int_{\mathbf{A}}^{\mathbf{B}} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = f(\mathbf{B}) - f(\mathbf{A}) \quad \text{where } \mathbf{F} = \operatorname{grad} f$

Exact: $\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$

Double integral: $\iint_R f(x,y) dx dy = \int_a^b \left[\int_{g(x)}^{h(x)} f(x,y) dy\right] dx = \int_c^d \left[\int_{p(y)}^{q(y)} f(x,y) dx\right] dy$

Change of variable: $\iint_R f(x,y) dx dy = \iint_{R'} f(x(u,v), y(u,v)) \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} du dv$