ERG2011A Tutorial 8: Laplace Transform

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1 Domains and Transform

- We can represent a function in two ways: e.g.

$$\sin t = \cos(t - \frac{\pi}{2})$$

- But we can represent it in a more complicated form: Fourier Transform, and Laplace Transform

1.1 Function of Functions

- Laplace transform is a function of functions, i.e.
  - Input: a function in t
  - Output: a function in s
  - For nearly any function in t, we can find an unique corresponding function in s
  - If we get the output, we can revert and get back the input (may differ by a constant)
- Transform:

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st} f(t) dt \\ \mathcal{L}^{-1}\{F(s)\} &= \lim_{R\to\infty} \frac{1}{2\pi i} \int_{\beta - iR}^{\beta + iR} e^{st} F(s) ds\end{aligned}$$

  - In Laplace transform, the s is a complex number, so as the limits of integration in $\mathcal{L}^{-1}$.
  - Laplace transform exists if the integral exists, i.e. $f(x) \le O(e^{-kt}) \quad \exists k \in \mathbb{R}$
  - Proof of them requires knowledge in complex analysis
- Hence, we prefer not to evaluate the integration directly
  - Look up the table, please!

$f(t)$	$\mathcal{L}(f)$	$f(t)$	$\mathcal{L}(f)$
1	$\frac{1}{s}$	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
t	$\frac{1}{s^2}$	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
t^2	$\frac{2}{s^3}$	$\cosh at$	$\frac{s}{s^2-a^2}$
$t^n \ (n \in \mathbb{N})$	$\frac{n!}{s^{n+1}}$	$\sinh at$	$\frac{a}{s^2-a^2}$
$t^a \ (a > 0)$	$\frac{\Gamma(a+1)}{s^{a+1}} = \frac{1}{s^{a+1}} \int_0^\infty e^{-x} x^a dx$	$e^{at} \cos \omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
e^{at}	$\frac{1}{s-a}$	$e^{at} \sin \omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$

1.2 Properties of Laplace Transform

- Linearity

$$\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

- Shifting

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

- Differentiation of function

$$\begin{aligned}\mathcal{L}\{f'(t)\} &= s\mathcal{L}\{f(t)\} - f(0)\\ \mathcal{L}\{\frac{d^n}{dt^n}f(t)\} &= s^n\mathcal{L}\{f(t)\} - \sum_{k=1}^{n}\left(s^{n-k}\,\frac{d^{k-1}f}{dt^{k-1}}\bigg|_{t=0}\right)\\ &= s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - s^1 f^{(n-2)}(0) - f^{(n-1)}(0)\end{aligned}$$

- Integration of function

$$\mathcal{L}\{\int_0^t f(\tau)d\tau\} = \frac{1}{s}\mathcal{L}\{f(t)\}$$

- Differentiation of transform

$$\mathcal{L}\{-tf(t)\} = \frac{d}{ds}\mathcal{L}\{f(t)\}$$

- Integration of transform

$$\mathcal{L}\{\frac{1}{t}f(t)\} = \int_s^\infty \mathcal{L}\{f(t)\}\ ds$$

- Convolution:

$$\mathcal{L}\{\int_0^t f(\tau)g(t-\tau)d\tau\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$$

  - How do convolution means?
  - Probability that tossing two identical dice gives a sum of 10 points
    1. If die A gives k, die B should give $10-k$ in order to have sum of 10
    2. k can be anything between 1 and 6
    3. The probability is $\sum_{k=1}^{6} p(k)p(10-k)$
  - Probability that queueing time p and buying lunch in Franklin Canteen q together cost you time t is
    1. If the queueing time is τ, the time you spend in dealing with the cashier is $t-\tau$ in order to have the total time spent$=t$
    2. τ can be anything between 0 and t
    3. The probability is $\sum_{\tau} p(\tau)q(1-\tau)$
    4. Because the time is continuous, we use integration: $\int_0^t p(\tau)q(t-\tau)d\tau$

1.3 Summary of transform properties

$f(t)$	$F(s)$
$af(t)+bg(t)$	$aF(s)+bG(s)$
$e^{at}f(t)$	$F(s-a)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$\frac{d^n}{dt^n}f(t)$	$s^nF(s)-\sum_{k=1}^{n}\left(s^{n-k}f^{(k-1)}(0)\right)$
$\int_0^t f(\tau)d\tau$	$\frac{1}{s}F(s)$
$-tf(t)$	$F'(s)$
$\frac{1}{t}f(t)$	$\int_s^\infty F(\sigma)d\sigma$
$f(t)*g(t)$	$F(s)G(s)$

1.4 Examples:

- Problem Set 5.1, Question 2: Find $\mathcal{L}\{a+bt+ct^2\}$

$$\begin{aligned}\mathcal{L}\{a+bt+ct^2\} &= a\mathcal{L}\{1\}+b\mathcal{L}\{t\}+c\mathcal{L}\{t^2\}\\ &= \frac{a}{s}+\frac{b}{s^2}+\frac{2c}{s^3}\end{aligned}$$

- Problem Set 5.1, Question 28: Find $\mathcal{L}^{-1}\{\frac{2s^3}{s^4-1}\}$

$$\begin{aligned}\frac{2s^3}{s^4-1} &= \frac{2s^3}{(s+1)(s-1)(s^2+1)}\\ &= \frac{1/2}{s+1}+\frac{1/2}{s-1}+\frac{s}{s^2+1} \qquad \Leftarrow \text{represent in partial fractions}\\ \mathcal{L}^{-1}\{\frac{2s^3}{s^4-1}\} &= \frac{1}{2}e^{-t}+\frac{1}{2}e^{t}+\cos t\end{aligned}$$

2 Use of Laplace Transform

2.1 Solving Differetial Equations with Initial Value Problems

- Because Laplace transform will cause differentaion and integration of $f(t)$ becomes algebraic expressions of $F(s)$, we can use it to simplify differential equations
- Way of thinking:

  1. Turn differential equations into subsidiary equations by Laplace transform
  2. Solve the subsidiary equations algebraically
  3. Do inverse transform on the solution of subsidiary equation
  4. Solution to the differential equation is obtained
  5. Check to make sure you did right

- Example: Problem Set 5.2, Question 4: $y'' - y' - 2y = 0, \quad y(0) = 8, \quad y'(0) = 7$

$$\begin{aligned}
y'' - y' - 2y &= 0 \\
\therefore \quad \mathcal{L}\{y'' - y' - 2y\} &= \mathcal{L}\{0\} \\
\left[s^2Y(s) - sy(0) - y'(0)\right] - [sY(s) - y(0)] - 2Y(s) &= 0 \\
(s^2 - s - 2)Y(s) - sy(0) - y'(0) + y(0) &= 0 \\
(s^2 - s - 2)Y(s) - 8s - 7 + 8 &= 0 \\
(s^2 - s - 2)Y(s) - 8s + 1 &= 0 \\
Y(s) &= \frac{8s - 1}{s^2 - s - 2} \\
&= \frac{5}{s - 2} + \frac{3}{s + 1} \\
\therefore \quad y(t) &= 5\mathcal{L}^{-1}\{\frac{1}{s - 2}\} + 3\mathcal{L}^{-1}\{\frac{1}{s + 1}\} \\
&= 5e^{2t} + 3e^{-t}
\end{aligned}$$

$$\begin{aligned}
\text{Verify:} \quad y(0) &= 5 + 3 = 8 \\
y'(t) &= 10e^{2t} - 3e^{-t} \\
\therefore \quad y'(0) &= 10 - 3 = 7 \\
y''(t) &= 20e^{2t} + 3e^{-t} \\
y'' - y' - 2y &= 20e^{2t} + 3e^{-t} - 10e^{2t} + 3e^{-t} - 10e^{2t} - 6e^{-t} \\
&= 0
\end{aligned}$$

- Example: Problem Set 5.2, Question 6: $y'' + y = 2\cos t, \quad y(0) = 3, \quad y'(0) = 4$

$$\begin{aligned}
y'' + y &= 2\cos t \\
\therefore \quad s^2Y(s) - sy(0) - y'(0) + Y(s) &= 2 \cdot \frac{s}{s^2+1} \\
(s^2+1)Y(s) - 3s - 4 &= \frac{2s}{s^2+1} \\
(s^2+1)^2Y(s) - (3s+4)(s^2+1) &= 2s \\
(s^2+1)^2Y(s) - (3s^3+3s+4s^2+4) &= 2s \\
(s^2+1)^2Y(s) &= 3s^3+4s^2+5s+4 \\
Y(s) &= \frac{3s^3+4s^2+5s+4}{(s^2+1)^2} \\
&= \frac{2s}{(s^2+1)^2} + \frac{3s+4}{s^2+1} \\
&= \frac{s}{s^2+1} \cdot \frac{2}{s^2+1} + \frac{3s}{s^2+1} + \frac{4}{s^2+1} \\
\therefore \quad y(t) &= 2\,(\cos t \ * \ \sin t) + 3\cos t + 4\sin t \\
&= \int_0^t 2\cos\tau \sin(t-\tau)d\tau + 3\cos t + 4\sin t \\
\text{By Product-to-sum formula,} \quad 2\cos\alpha\sin\beta &= \sin(\alpha+\beta) - \sin(\alpha-\beta) \\
\therefore \quad y(t) &= \int_0^t (\sin t - \sin(2\tau - t))\,d\tau + 3\cos t + 4\sin t \\
&= \int_0^t \sin t d\tau - \frac{1}{2}\int_{-t}^t \sin(2\tau - t)d(2\tau - t) + 3\cos t + 4\sin t \\
&= t\sin t - \frac{1}{2}\left[-\cos u\right]_{-t}^t + 3\cos t + 4\sin t \\
&= t\sin t - \frac{1}{2}\left[0\right] + 3\cos t + 4\sin t \\
&= t\sin t + 3\cos t + 4\sin t \\
&= 3\cos t + (t+4)\sin t
\end{aligned}$$

$$\begin{aligned}
\text{Verify:} \quad y(0) &= 3 \\
y' &= -3\sin t + (t+4)\cos t + \sin t \\
&= -2\sin t + (t+4)\cos t \\
\therefore y'(0) &= 4 \\
y'' &= -2\cos t - (t+4)\sin t + \cos t \\
&= -\cos t - (t+4)\sin t \\
\therefore \quad y'' + y &= -\cos t - (t+4)\sin t + 3\cos t + (t+4)\sin t \\
&= 2\cos t
\end{aligned}$$

- Example: Problem Set 5.2, Question 8: $y'' + 0.04y = 0.02t^2, \quad y(0) = -25, \quad y'(0) = 0$

$$\begin{aligned}
y'' + 0.04y &= 0.02t^2 \\
\therefore \quad s^2Y(s) - sy(0) - y'(0) + 0.04Y(s) &= 0.02 \cdot \frac{2}{s^3} \\
(s^2 + 0.04)Y(s) + 25s &= \frac{0.04}{s^3} \\
(s^5 + 0.04s^3)Y(s) + 25s^4 &= 0.04 \\
Y(s) &= \frac{0.04 - 25s^4}{s^5 + 0.04s^3} \\
&= \frac{1 - 625s^4}{25s^5 + s^3} \\
&= \frac{1}{s^3} - \frac{25}{s} \\
&= \frac{1}{2} \cdot \frac{2}{s^3} - 25 \cdot \frac{1}{s} \\
\therefore \quad y(t) &= \frac{1}{2}t^2 - 25
\end{aligned}$$

$$\begin{aligned}
\text{Verify:} \quad y(0) &= -25 \\
y'(t) &= t \\
\therefore \quad y'(0) &= 0 \\
y''(t) &= 1 \\
\therefore \quad y'' + 0.04y &= 1 + \frac{1}{25}\left(\frac{1}{2}t^2 - 25\right) \\
&= 1 + \frac{1}{50}t^2 - 1 \\
&= \frac{1}{50}t^2 = 0.02t^2
\end{aligned}$$

2.2 Solving integral equation

- Especially those involves convolution, read lecture notes section V

2.3 Solving systems of differential equation

- Using Laplace transform, we convert all differentiations into algebraic expression of $\mathcal{L}(y)$, hence the system of differential equations is same as system of linear equations
- Use Crammer's rule or other methods to solve it!
- See lecture notes section VII

2.4 Studying systems

- The real use in Engineering: how those differential equations comes out
- Department of Information Engineering offers a course IEG2051 about this