ERG2011A Tutorial 9: Step and Impulse Functions; Fourier Series

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20th November 2004

1 Step Functions

1.1 Definition

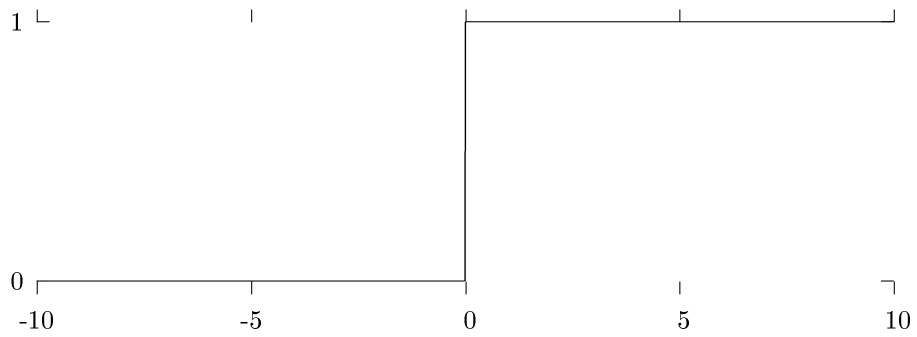

Figure 1: Unit step function

- Step function is: $u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$
- In mathematica, you can use `UnitStep[t]` to represent this.
- $u(t-a)$ means it turns from 0 to 1 when $t =$
- For a given function $f(t)$, the multiplication of $u(t-a)$ means we turn on the function at time $t =$
- Example: $f(t) = \sin t$; $u(t-\pi)f(t)$:

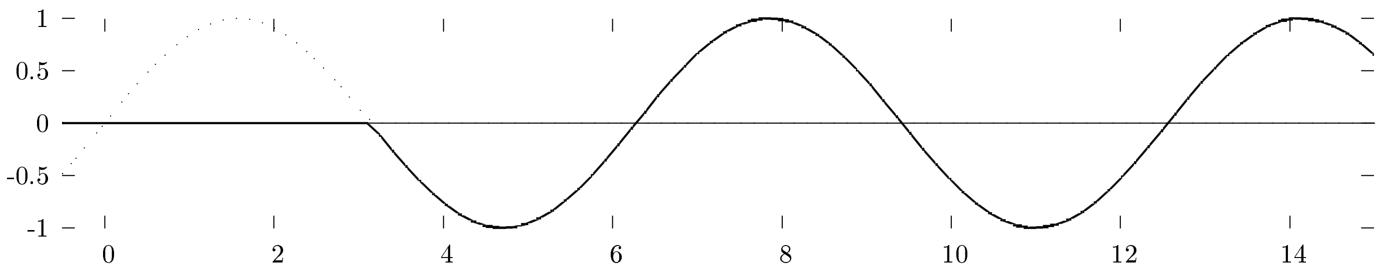

Figure 2: $y = u(x-\pi)\sin x$

- Essentially, $u(t-a)f(t) = \begin{cases} f(t) & t \geq a \\ 0 & t < a \end{cases}$

1.2 Laplace transform

- Laplace transform:

$$\begin{aligned}\mathcal{L}\{f(t-a)u(t-a)\} &= e^{-as}F(s)\\ \mathcal{L}^{-1}\{e^{-as}F(s)\} &= f(t-a)u(t-a)\end{aligned}$$

- For just $u(t-a)$, we can assume $f(t)=1$ and obtains

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$

- Example: Problem Set 5.3 Question 7, $\mathcal{L}\{4u(t-\pi)\cos t\}$

$$\begin{aligned}\mathcal{L}\{4u(t-\pi)\cos t\} &= \mathcal{L}\{\quad 4u(t-\pi)\cos(\qquad\qquad)\}\\ &= -4\mathcal{L}\{u(t-\pi)\cos(t-\pi)\}\\ &= -4e^{-\pi s}\mathcal{L}\{\cos(t)\}\\ &= -4e^{-\pi s}\cdot(\frac{\qquad}{\qquad})\\ &= \frac{4se^{-\pi s}}{1-s^2}\end{aligned}$$

- Example: Problem Set 5.3 Question 18, $\mathcal{L}^{-1}\{e^{-2\pi s}/(s^2+2s+2\}$

$$\begin{aligned}\mathcal{L}^{-1}\{\frac{e^{-2\pi s}}{s^2+2s+2}\} &= \mathcal{L}^{-1}\{e^{-2\pi s}\frac{1}{s^2+2s+2}\}\\ &= \mathcal{L}^{-1}\{e^{-2\pi s}\frac{1}{(s+1)^2+1}\}\\ &= u(t-2\pi)\cdot\mathcal{L}^{-1}\{\frac{1}{(s+1)^2+1}\}\Big|_{t\to}\\ &= u(t-2\pi)\cdot\mathcal{L}^{-1}\{\frac{1}{[s-(\qquad)]^2+1}\}\Big|_{t\to t-2\pi}\\ &= u(t-2\pi)\cdot e\quad\mathcal{L}^{-1}\{\frac{1}{s^2+1}\}\Big|_{t\to t-2\pi}\\ &= u(t-2\pi)\cdot e^{-t}\sin t\big|_{t\to t-2\pi}\\ &= u(t-2\pi)e^{-(t-2\pi)}\sin(t-2\pi)\\ &= u(t-2\pi)e^{-(t-2\pi)}\sin t\end{aligned}$$

2 Impulse function

2.1 Definition

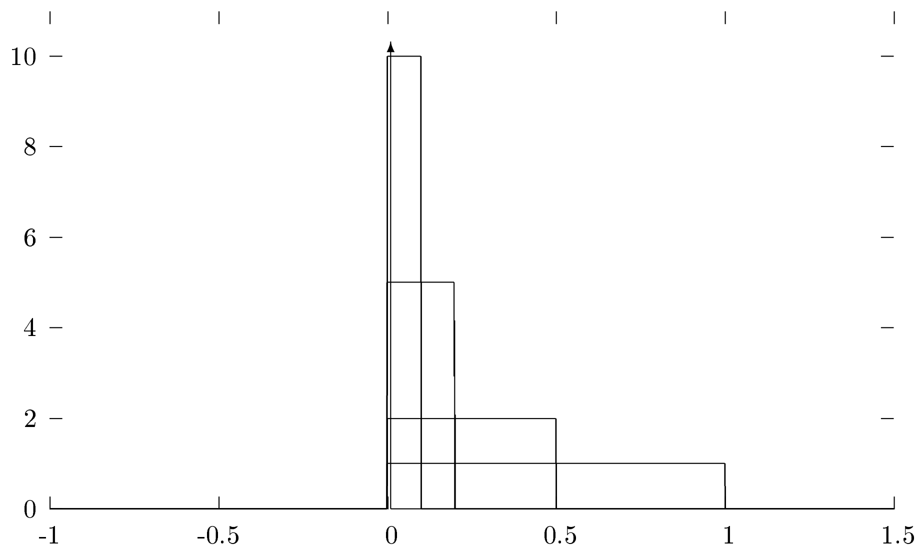

Figure 3: Impulse function

- Impulse function is: $\delta(t) = \lim_{\tau \to 0} \begin{cases} 1/\tau & 0 \le t \le \tau \\ 0 & \text{otherwise} \end{cases}$
- By definition, $\int_{-\infty}^{\infty} \delta(t) dt = 1$
- It is also called the "Dirac Delta Function" or "Unit Impulse Function"
- In mathematica, you can use `DiracDelta[t]` to represent this
- Graphically, we usually write an up arrow at $t = 0$ to represent this
- Physically, an impulse means to give a short hit as an input at $t = 0$
- In $\delta(t - a)$, it means the hit is at $t =$

2.2 Laplace Transform

- Laplace transform:

$$\begin{aligned} \mathcal{L}\{\delta(t-a)\} &= e^{-as} \\ \mathcal{L}^{-1}\{e^{-as}\} &= \delta(t-a) \end{aligned}$$

- Please note that:
  - By definition, $\int_{-\infty}^{t} \delta(\tau) d\tau = u(t)$
  - By convolution property of Laplace transform: $\int_{0}^{t} f(\tau)\delta(t-\tau) d\tau = u(t-\tau)f(t-\tau)$
    - ∗ You can check this by simple reasoning and sketching!

3 Fourier Series

- Completely different thing!

3.1 Periodicity with period 2π

- Many things are periodic
- In high school physics, we learnt about superposition of two sine waves with different frequency causes beat
- Can I give you a superposition of some sine waves and you tell me how does it constitutes?
  - Fourier series prepresentation of a periodic function
- Claim: Every periodic function is a superposition of (sometimes infinitely many) sine and cosine waves!
- Fourier series of a function with period 2π: $f(x) = a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$
  - Period 2π means: $f(x+\qquad) = f(x)$ for all x
- Finding Fourier coefficients:

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \\ a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kxdx \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kxdx \end{aligned}$$

3.2 Periodicity with any period

- Assume $g(x)$ is a function with period 2π, then the function $f(x) = g(2\pi x/p)$ has period p:

$$\begin{aligned} g(x+2\pi) &= g(x) \\ f(x+p) &= g((2\pi x + 2\pi p)/p) \\ &= g(2\pi x/p + 2\pi) \\ &= g(2\pi x/p) \\ &= f(x) \end{aligned}$$

- Fourier series of a function with period $p = 2L$:

$$\begin{aligned} f(x) &= a_0 + \sum_{k=1}^{\infty}\left(a_k \cos\frac{k\pi}{L}x + b_k \sin\frac{k\pi}{K}x\right) \\ a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \\ a_k &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{k\pi}{L}xdx \\ b_k &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{k\pi}{L}xdx \end{aligned}$$

- Example: Problem Set 10.3 Question 4

  Find the Fourier Series of the periodic function $f(x) = |x| \quad (-2 < x < 2), \quad p = 2L = 4$

$$
\begin{aligned}
f(x) &= a_0 + \sum_{k=1}^{\infty} \left(a_k \cos \frac{k\pi}{L} x + b_k \sin \frac{k\pi}{K} x \right) \\
a_0 &= \frac{1}{2L} \int_{-L}^{L} f(x) dx \\
&= \frac{1}{4} \int_{-2}^{2} |x| dx \\
&= \frac{1}{4} \int_{-2}^{0} (-x) dx + \frac{1}{4} \int_{0}^{2} x dx \\
&= \frac{1}{4} \int_{0}^{2} (-x) d(-x) + \frac{1}{4} \int_{0}^{2} x dx \\
&= \frac{1}{2} \int_{0}^{2} x dx \\
&= [x^2]_0^2 \\
&= 4 \\
a_k &= \frac{1}{2} \int_{-2}^{2} |x| \cos \frac{k\pi}{2} x dx \\
&= -\frac{1}{2} \int_{-2}^{0} x \cos \frac{k\pi}{2} x dx + \frac{1}{2} \int_{0}^{2} x \cos \frac{k\pi}{2} x dx \\
&= \frac{1}{2} \int_{0}^{-2} x \cos \frac{k\pi}{2} x dx + \frac{1}{2} \int_{0}^{2} x \cos \frac{k\pi}{2} x dx \\
&= \frac{1}{2} \int_{0}^{2} (-x) \cos \frac{k\pi}{2} (-x) d(-x) + \frac{1}{2} \int_{0}^{2} x \cos \frac{k\pi}{2} x dx \\
&= \frac{1}{2} \int_{0}^{2} x \cos \frac{k\pi}{2} x dx + \frac{1}{2} \int_{0}^{2} x \cos \frac{k\pi}{2} x dx \\
&= \int_{0}^{2} x \cos \frac{k\pi}{2} x dx \\
&= \left[\frac{4}{k^2\pi^2} \cos \frac{k\pi}{2} x - \frac{2x}{k\pi} \sin \frac{k\pi}{2} x \right]_0^2 \\
&= \frac{4}{k^2\pi^2} \cos k\pi - \frac{4}{k\pi} \sin k\pi - \frac{4}{k^2\pi^2} \\
&= \frac{4}{k^2\pi^2} (\cos k\pi - 1) \\
&= \frac{4}{k^2\pi^2} \left((-1)^k - 1 \right) \\
b_k &= \frac{1}{2} \int_{-2}^{2} |x| \sin \frac{k\pi}{2} x dx \\
&= -\frac{1}{2} \int_{-2}^{0} x \sin \frac{k\pi}{2} x dx + \frac{1}{2} \int_{0}^{2} x \sin \frac{k\pi}{2} x dx \\
&= \frac{1}{2} \int_{0}^{-2} x \sin \frac{k\pi}{2} x dx + \frac{1}{2} \int_{0}^{2} x \sin \frac{k\pi}{2} x dx \\
&= \frac{1}{2} \int_{0}^{2} (-x) \sin \frac{k\pi}{2} (-x) d(-x) + \frac{1}{2} \int_{0}^{2} x \sin \frac{k\pi}{2} x dx \\
&= -\frac{1}{2} \int_{0}^{2} x \sin \frac{k\pi}{2} x dx + \frac{1}{2} \int_{0}^{2} x \sin \frac{k\pi}{2} x dx \\
&= 0 \\
\therefore \quad f(x) &= 4 + \frac{4}{\pi^2} \sum_{k=1}^{\infty} \left(\frac{(-1)^k - 1}{k^2} \cos \frac{k\pi}{2} x \right)
\end{aligned}
$$

3.3 Even and Odd Functions

- Some functions are called even functions if they satisfy: $f(-x) = f(x)$
- Some functions are called odd functions if they satisfy: $f(-x) = -f(x)$
- Properties of even/odd functions:
  1. Sum of even functions is even
  2. Sum of two odd functions is even
  3. Product of two even/odd functions is even
  4. Product of an even function and an odd function is odd
  5. $\int_{-L}^{L} f_{\text{even}}(x)dx = 2\int_{0}^{L} f_{\text{even}}(x)dx$
  6. $\int_{-L}^{L} f_{\text{odd}}(x)dx = 0$
- In Fourier series representation, all odd functions have $a_k \equiv 0$

$$f_{\text{odd}}(x) = \sum_{k=1}^{\infty} \left(b_k \sin \frac{k\pi}{K} x \right)$$

- In Fourier series representation, all even functions have $b_k \equiv 0$

$$f_{\text{even}}(x) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos \frac{k\pi}{K} x \right)$$

- Further properties of Fourier series representation:
  - $f(x) = f_1(x) + f_2(x)$ then the Fourier series is the sum of every corresponding coefficients
  - $cf(x)$ has the Fourier series with each Fourier coefficients of $f(x)$ multiplied by c